Piecewise Linear Virtual Inputs/ Outputs in Interval DEA

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ABSTRACT

A recent development in data envelopment analysis (DEA) concerns the introduction of a piece-wise linear representation of the virtual inputs and/or outputs as a means to model situations where the marginal value of an output (input) is assumed to diminish (increase) as the output (input) increases. Currently, this approach is limited to crisp data sets. In this paper, the authors extend the piece-wise linear approach to interval DEA, i.e. to cases where the input/output data are only known to lie within intervals with given bounds. The authors also define appropriate interval segmentations to implement the piece-wise linear forms in conjunction with the interval bounds of the input/output data and the authors propose a new models, compliant with the interval DEA methodology. They finally illustrate their developments with an artificial data set.

Keywords: *Crisp Data Sets, Data Envelopment Analysis (DEA), Interval DEA, Marginal Value, Piece-Wise Linear Approach, Piecewise Linear Virtual Inputs/Outputs*

INTRODUCTION

Data envelopment analysis (DEA) is the leading technique for assessing the efficiency of decision making units (DMU) in the presence of multiple inputs and outputs. The two milestone DEA models, namely the CCR (Charnes et al., 1978) and the BCC (Banker et al., 1984) models have become standards in the literature of performance measurement. Recent applications of DEA include, among others, those of Mahdavi et al. (2008), Martin and Roman (2010), Pramodth et al. (2008) and Sufian (2010). The underlying mathematical instrument for performing the analysis is linear programming. Performing a typical DEA analysis means solving a series of linear programs, one for each DMU. Efficiency is measured in a bounded ratio scale by the fraction 'weighted output' to 'weighted input'. The inputs and outputs are assumed to be continuous positive variables and the weights are estimated through the associated linear program in favor of the evaluated unit so as to maximize its efficiency.

Focusing on the outputs, an output measure multiplied by the associated weight is called *virtual output*. The summation of the virtual outputs over all the output dimensions, called

DOI: 10.4018/joris.2013040103

total virtual output, forms the numerator of the efficiency ratio. A typical interpretation of the weights is that they represent marginal values of outputs. In this manner the virtual outputs can be conceived as *linear partial value functions* and the total virtual output as an overall additive *value function*. According to Dyson et al. (2001) the linearity assumption underlying the virtual outputs might be unjustifiable in cases where the marginal value of an output diminishes as the output increases. Recently, Cook and Zhu (2009) and Despotis et al. (2010), motivated by applications involving non-linear virtual outputs, proposed a piece-wise linear representation of the partial value functions as a means to model the situation where particular outputs exhibit diminishing returns. Despotis et al. (2010) showed that ordinary DEA models can be used to perform the efficiency assessments by appropriately introducing additional input/output dimensions in the original data set.

In this paper we extend the piece-wise linear approach in interval DEA to fit the case where the DEA efficiency assessments must be performed on the basis of input and/or output data that are only known to lie within intervals with given bounds (interval data). We reformulate the partial value functions (virtual inputs and outputs) by introducing additional input/output dimensions to obtain an augmented data set that will form the basis for interval efficiency assessments. The rest of the paper unfolds as follows. In the second section we revisit the piece-wise linear DEA models as applied on crisp data to present a simplified formulation, which will be the basis for our new developments. In the third section we provide a brief description of the interval DEA models proposed by Despotis and Smirlis (2002). In the fourth section we provide our main developments that extend the piece-wise linear DEA approach to interval DEA and we formulate appropriate models capable of estimating lower and upper bound efficiencies when inputs (outputs) exhibit increasing (diminishing) returns. In the fifth section we illustrate our new developments with an artificial data set. The paper ends with some concluding remarks.

DEA MODELS WITH NON-LINEAR PARTIAL VALUE FUNCTIONS

Consider the following input-oriented CCR DEA model (*multiplier* form) with n DMUs, m inputs and s outputs, where y_{rj} denotes the level of the output r (r =1,…, s) produced by the DMU j (j=1,…,n), x_{ij} denotes the level of the input i (i=1,…, m) consumed by the DMU j and the variables u=(u_r, r=1,…,s) and v=(v_i, i=1,…,m) are the unknown weights attached to the outputs and the inputs respectively:

$$\begin{aligned} &\max h_{j_0} = \sum_{r=1}^{s} u_r y_{rj_0} \\ &s.t. \\ &\sum_{i=1}^{m} v_i x_{ij_0} = 1 \\ &\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \;\; j = 1,...,n \\ &u_r, v_i \ge 0 \quad \forall r, i \end{aligned} \tag{1}$$

Model (1) estimates the relative efficiency h_{j_0} of the evaluated DMU j_0 and is solved repeatedly for every DMU $j, j = 1,...,n$. Let $U_r(y_{rj}), r = 1,...,s$ and $U_i(x_{ij}), i = 1,...,m$ denote the virtual outputs and inputs for unit j respectively. Then

$$U(Y_j) = \sum_{r=1}^{s} U_r(y_{rj}) = \sum_{r=1}^{s} u_r y_{rj} \tag{3}$$

and

$$U(X_j) = \sum_{i=1}^{m} U_i(x_{ij}) = \sum_{i=1}^{m} v_i x_{ij} \tag{4}$$

are the total virtual output and input respectively for unit j, which are linear functions of the weights.

Recently, Cook and Zhu (2009) and Despotis et al. (2010) relaxed the linearity assumption in DEA by introducing a piece-wise linear representation of the virtual inputs and outputs. Setting $y_r^{\min}$ and $y_r^{\max}$ at the levels of the lowest and the highest observed values of output r respectively, they split the interval $[y_r^{\min}, y_r^{\max}]$ into successive and non-overlapping segments by taking a number of breakpoints and then assigned a different weight for each segment. Restrictions on the weights are then imposed to drive the concavity or the convexity of the value functions.

For the clarity of the presentation, we simplify the models introduced in Despotis et al. (2010), by assuming only one breakpoint y_r^0 that splits the range of values of output r in two sub-intervals $[y_r^{\min}\ ,\ y_r^0],\ \ (y_r^0,\ y_r^{\max}]$. Figure 1 depicts a typical linear value function, as assumed in the original DEA model (line a), increasing returns beyond the threshold y_r^0 (line segment b) and diminishing returns beyond the threshold y_r^0 (line segment c).

On the basis of the above segmentation, the output value $y_{rj} \in [y_r^{\min}, y_r^{\max}]$ of any unit j is decomposed in two parts and is expressed as $y_{rj} = \delta_{rj}^1 + \delta_{rj}^2$, where:

$$\delta_{rj}^1 = \begin{cases} y_{rj} \ \text{ if } y_{rj} \in [y_r^{\min},\ y_r^o] \\ y_r^o \ \text{ if } y_{rj} \in (y_r^o,\ y_r^{\max}] \end{cases},$$

$$\delta_{rj}^2 = \begin{cases} 0 \qquad\quad \text{ if } y_{rj} \in [y_r^{\min},\ y_r^o] \\ y_{rj} - y_r^o \ \text{ if } y_{rj} \in (y_r^o\ , y_r^{\max}] \end{cases}$$

In this manner, the partial value $U_r(y_{rj})$ is modeled in a piece-wise linear form:

$$U_r(y_{rj}) = u_{r1}\delta_{rj}^1 + u_{r2}\delta_{rj}^2 \qquad (5)$$

where u_{r1} and u_{r2} are the distinct weights associated with the two sub-intervals.

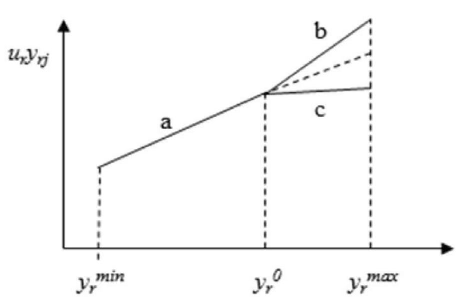
In general, the nonlinearity assumption is applicable or desirable for particular outputs only (*nonlinear outputs)*, with the rest of them complying with the linearity assumption. Without loss of generality, we assume that the first d ($d<s$) outputs are linear and the rest of them (i.e. for $r = d+1,\dots,s$) nonlinear. Then the total virtual output (3) takes the following form:

$$U(Y_j) = \sum_{r=1}^{d} u_r y_{rj} + \sum_{r=d+1}^{s} (u_{r1}\delta_{rj}^1 + u_{r2}\delta_{rj}^2)$$

The virtual inputs are modeled analogously. Indeed, if $[x_i^{\min}, x_i^{\max}]$ is the interval defined by the minimum and the maximum values of input i and x_i^0 is the breakpoint that splits this interval in two segments, the input value $x_{ij} \in [x_i^{\min}, x_i^{\max}]$ of any unit j is decomposed in two parts $x_{ij} = \gamma_{ij}^1 + \gamma_{ij}^2$ where:

$$\gamma_{ij}^1 = \begin{cases} x_{ij} \ \text{ if } x_{ij} \in [x_i^{\min},\ x_i^o] \\ x_i^o \ \text{ if } x_{ij} \in (x_i^o,\ x_i^{\max}] \end{cases},$$

Figure 1. Piece-wise linear value function for output r



$$\gamma_{ij}^2 = \begin{cases} 0 & \text{if } x_{ij} \in [x_i^{\min},\, x_i^o] \\ x_{ij} - x_i^o & \text{if } x_{ij} \in (x_i^o,\, x_i^{\max}] \end{cases}$$

and the virtual input $U_i(x_{ij})$ is modeled as a piece-wise linear function:

$$U_i(x_{ij}) = v_{i1}\gamma_{rj}^1 + v_{i2}\gamma_{rj}^2 \quad (6)$$

where v_{i1} and v_{i2} are the input weights associated with the two sub-intervals. Then the total virtual input (4) is given by the following equation:

$$U(X_j) = \sum_{i=1}^{t} v_i x_{ij} + \sum_{i=t+1}^{m} (v_{i1}\gamma_{ij}^1 + v_{i2}\gamma_{ij}^2)$$

where the first t inputs are assumed linear and the rest of them non-linear.

Imposing the homogeneous restrictions $u_{r1} - c_r u_{r2} \geq 0$ ($c_r > 1$) on the weights u_{r1} and u_{r2}, the value function (5) is restricted to be concave. Similarly, the relations $-v_{i1} + v_{i2} z_i \geq 0$ ($0 < z_i < 1$), on the weights v_{i1} and v_{i2}, restrict the value function (6) to be convex.

The formulations presented above transform the original data set into an augmented data set by decomposing each one of the non-linear inputs and outputs in two auxiliary linear inputs and linear outputs respectively. This transformation allows us to perform the efficiency assessments without drawing away from the grounds of the standard DEA methodology. The model (7) below is a piece-wise linear DEA model with weight restrictions imposing concave value functions for outputs and convex value functions for inputs. As the inputs are in the denominator of the efficiency ratio, convex value functions penalize the excess inputs.

$$\begin{aligned}
&\max h_{j_0} = U(Y_{j_0}) \\
&s.t. \\
&U(X_{j_0}) = 1 \\
&U(Y_j) - U(X_j) \leq 0 \quad j = 1, \ldots, n \\
&U(Y_j) = \sum_{r=1}^{d} u_r y_{rj} + \sum_{r=d+1}^{s} (u_{r1}\delta_{rj}^1 + u_{r2}\delta_{rj}^2) \\
&U(X_j) = \sum_{i=1}^{t} v_i x_{ij} + \sum_{i=t+1}^{m} (v_{i1}\gamma_{ij}^1 + v_{i2}\gamma_{ij}^2) \\
&u_{r1} - c_r u_{r2} \geq 0,\ r = d+1, \ldots, s \quad (c_r > 1) \\
&-v_{i1} + v_{i2} z_i \geq 0,\ i = t+1, \ldots, m \quad (0 < z_i < 1) \\
&u_r, v_i \geq 0 \quad r = 1, \ldots, d,\ i = 1, \ldots t \\
&u_{r1}, u_{r2}, v_{i1}, v_{i2} \geq 0 \quad r = d+1, \ldots, s,\ i = t+1, \ldots, m
\end{aligned} \quad (7)$$

AN INTERVAL DEA PRIMER

In interval DEA, the inputs and the outputs are only known to lie within intervals with given positive lower and upper bounds, i.e. $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{ij} \in [y_{rj}^L, y_{rj}^U]$. Thus the original DEA model (1) becomes non-linear as the inputs and the outputs are treated as bounded variables rather than as constants. Since the seminal paper of Cooper et al. (1999), a number of approaches have been proposed to deal with imprecise data in DEA- a broader spectrum of data variants such as crisp, interval and ordinal data (e.g. Despotis & Smirlis, 2002; Entani et al., 2002; Zhu, 2003; Wang et al., 2005; Jahanshahloo et al., 2009; Kao, 2006; Shokouhi et al., 2010, among others).

Despotis and Smirlis (2002) introduced the notion of interval efficiency assessments in DEA that is, for each DMU, a lower and an upper bound of the efficiency scores is estimated. They transformed the aforementioned nonlinear DEA model to a linear equivalent by applying simple variable transformations. Then they showed that the upper efficiency bound $h_{j_0}^U$ and the lower efficiency bound $h_{j_0}^L$ are obtained by the following couple of standard DEA models respectively:

Upper efficiency bound

$$
\begin{aligned}
&\max h_{j_0}^U = \sum\nolimits_{r=1}^{s} u_r y_{rj_0}^U \\
&s.t. \\
&\sum\nolimits_{i=1}^{m} v_i x_{ij_0}^L = 1 \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj_0}^U - \sum\nolimits_{i=1}^{m} v_i x_{ij_0}^L \le 0 \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj}^L - \sum\nolimits_{i=1}^{m} v_i x_{ij}^U \le 0, j = 1,..,n; j \ne j_0 \\
&u_r, v_i \ge 0 \quad \forall r, i
\end{aligned} \tag{8}
$$

Lower efficiency bound

$$
\begin{aligned}
&\max h_{j_0}^L = \sum\nolimits_{r=1}^{s} u_r y_{rj_0}^L \\
&s.t. \\
&\sum\nolimits_{i=1}^{m} v_i x_{ij_0}^U = 1 \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj_0}^L - \sum\nolimits_{i=1}^{m} v_i x_{ij_0}^U \le 0 \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj}^U - \sum\nolimits_{i=1}^{m} v_i x_{ij}^L \le 0, j = 1,...,n; j \ne j_0 \\
&u_r, v_i \ge 0 \quad \forall r, i
\end{aligned} \tag{9}
$$

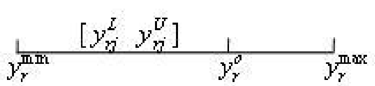
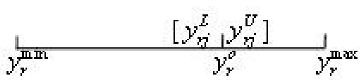
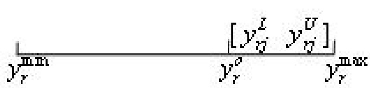
Models (7)-(9) above are the basis for our new developments in the next section.

MODELING PIECEWISE LINEAR VIRTUAL INPUTS/OUTPUTS IN INTERVAL DEA

To model the piecewise linear virtual outputs when interval data are considered, let us introduce the following notation for an interval output measure r: $y_r^{\min} = \min\{y_{rj}^L, j = 1,..,n\}$ and $y_r^{\max} = \max\{y_{rj}^U, j = 1,..,n\}$ denote respectively the minimum lower bound and the maximum upper bound over the interval observations of output r for all DMUs. Let $y_r^o \in [y_r^{\min}, y_r^{\max}]$ be a given threshold value beyond which the output r exhibits diminishing returns. The threshold y_r^o splits the interval $[y_r^{\min}, y_r^{\max}]$ of output r in two sub-intervals $[y_r^{\min}, y_r^0]$ and $(y_r^o, y_r^{\max}]$. The value function for output r is then assumed linear in each sub-interval and piecewise linear over $[y_r^{\min}, y_r^{\max}]$. The relative position of the interval output $[y_{rj}^L, y_{rj}^U]$ of any unit j with regard to the threshold value y_r^o is depicted in Table 1.

Employing interval arithmetic, the interval $[y_{rj}^L, y_{rj}^U]$ can then be decomposed in two sub-intervals as follows:

Table 1. Relative positions of $[y_{rj}^L, y_{rj}^U]$ with regard to the threshold y_r^o

Case	Position relative to the threshold value y_r^o	Condition
(i)		$y_{rj}^U < y_r^0$
(ii)		$y_{rj}^L \le y_r^0 \le y_{rj}^U$
(iii)		$y_r^0 < y_{rj}^L$

$$[y_{rj}^L, y_{rj}^U] = \begin{cases} [y_{rj}^L, y_{rj}^U] + [0,0] & \text{if } y_{rj}^U < y_r^0 \\ [y_{rj}^L, y_r^0] + [0, y_{rj}^U - y_r^0] & \text{if } y_{rj}^L \le y_r^0 \le y_{rj}^U \\ [y_r^0, y_r^0] + [y_{rj}^L - y_r^0, y_{rj}^U - y_r^0] & \text{if } y_r^0 < y_{rj}^L \end{cases}$$

On the basis of the above segmentation, each output interval $[y_{rj}^L, y_{rj}^U]$ is decomposed in two intervals $[\delta 1_{rj}^L, \delta 1_{rj}^U]$ and $[\delta 2_{rj}^L, \delta 2_{rj}^U]$, presented in Box 1.

In this manner the original interval data set is transformed to an augmented data set by decomposing each interval output measure into a couple of auxiliary interval output measures, to which distinct weights u_{r1} and u_{r2} are assigned.

The treatment of interval inputs is quite similar. Indeed, if $x_i^o \in [x_i^{\min}, x_i^{\max}]$ is a specified threshold within the range of input *i,* each interval $[x_{xj}^L, x_{xj}^U]$ is decomposed in two intervals $[\gamma 1_{ij}^L, {}^3 1_{ij}^U]$ and $[\gamma 2_{ij}^L, \gamma 2_{ij}^U]$, presented in Box 2.

Assuming that the first *d* outputs are linear and the rest *s-d* non-linear, we introduce the segmentation defined in (10) to the total virtual outputs in models (8) and (9), which now take the form:

$$U(Y_j^L) = \sum_{r=1}^{d} u_r y_{rj}^L + \sum_{r=d+1}^{s} u_{r1} \delta 1_{rj}^L + \sum_{r=d+1}^{s} u_{r2} \delta 2_{rj}^L$$
$$U(Y_j^U) = \sum_{r=1}^{d} u_r y_{rj}^U + \sum_{r=d+1}^{s} u_{r1} \delta 1_{rj}^U + \sum_{r=d+1}^{s} u_{r2} \delta 2_{rj}^U \quad (12)$$

Similarly, the total virtual inputs in (8) and (9) take the form (13) as follows:

$$U(X_j^L) = \sum_{i=1}^{t} v_i x_{ij}^L + \sum_{i=t+1}^{m} v_{i1} \gamma 1_{ij}^L + \sum_{i=t+1}^{m} v_{i2} \gamma 2_{ij}^L$$
$$U(X_j^U) = \sum_{i=1}^{t} v_i x_{ij}^U + \sum_{i=t+1}^{m} v_{i1} \gamma 1_{ij}^U + \sum_{i=t+1}^{m} v_{i2} \gamma 2_{ij}^U \quad (13)$$

Box 1.

$$\delta 1_{rj}^L = \begin{cases} y_{rj}^L & \text{if } y_{rj}^U < y_r^0 \\ y_{rj}^L & \text{if } y_{rj}^L \le y_r^0 \le y_{rj}^U \\ y_r^0 & \text{if } y_r^0 < y_{rj}^L \end{cases} \qquad \delta 1_{rj}^U = \begin{cases} y_{rj}^U & \text{if } y_{rj}^U < y_r^0 \\ y_r^0 & \text{if } y_{rj}^L \le y_r^0 \le y_{rj}^U \\ y_r^0 & \text{if } y_r^0 < y_{rj}^L \end{cases}$$
$$\delta 2_{rj}^L = \begin{cases} 0 & \text{if } y_{rj}^U < y_r^0 \\ 0 & \text{if } y_{rj}^L \le y_r^0 \le y_{rj}^U \\ y_{rj}^L - y_r^0 & \text{if } y_r^0 < y_{rj}^L \end{cases} \qquad \delta 2_{rj}^U = \begin{cases} 0 & \text{if } y_{rj}^U < y_r^0 \\ y_{rj}^U - y_r^0 & \text{if } y_{rj}^L \le y_r^0 \le y_{rj}^U \\ y_{rj}^U - y_r^0 & \text{if } y_r^0 < y_{rj}^L \end{cases} \quad (10)$$

Box 2.

$$\gamma 1_{ij}^L = \begin{cases} x_{ij}^L & \text{if } x_{ij}^U < x_i^0 \\ x_{ij}^L & \text{if } x_{ij}^L \le x_i^0 \le x_{ij}^U \\ x_i^0 & \text{if } x_i^0 < x_{ij}^L \end{cases} \qquad \gamma 1_{ij}^U = \begin{cases} x_{ij}^U & \text{if } x_{ij}^U < x_i^0 \\ x_i^0 & \text{if } x_{ij}^L \le x_i^0 \le x_{ij}^U \\ x_i^0 & \text{if } x_i^0 < x_{ij}^L \end{cases}$$
$$\gamma 2_{ij}^L = \begin{cases} 0 & \text{if } x_{ij}^U < x_i^0 \\ 0 & \text{if } x_{ij}^L \le x_i^0 \le x_{ij}^U \\ x_{ij}^L - x_i^0 & \text{if } x_i^0 < x_{ij}^L \end{cases} \qquad \gamma 2_{ij}^U = \begin{cases} 0 & \text{if } x_{ij}^U < x_i^0 \\ x_{ij}^U - x_i^0 & \text{if } x_{ij}^L \le x_i^0 \le x_{ij}^U \\ x_{ij}^U - x_i^0 & \text{if } x_i^0 < x_{ij}^L \end{cases} \quad (11)$$

where the first t inputs are assumed linear and the rest m-t non-linear.

Introducing (12) and (13), the DEA models for estimating the upper and lower efficiency bounds in the presence of concave virtual outputs and convex virtual inputs are as follows:

Upper efficiency bound $h_{j_0}^U$

$$\begin{aligned}
&\max h_{j_0}^U = U(Y_{j_0}^U)\\
&s.t.\\
&U(X_{j_0}^L) = 1\\
&U(Y_{j_0}^U) - U(X_{j_0}^L) \le 0\\
&U(Y_j^L) - U(X_j^U) \le 0 \quad j = 1,\dots,n, \quad j \ne j_0\\
&U(X_j^L) = \sum_{i=1}^{t} v_i x_{ij}^L + \sum_{i=t+1}^{m} v_{i1}\gamma 1_{ij}^L + \sum_{i=t+1}^{m} v_{i2}\gamma 2_{ij}^L\\
&U(X_j^U) = \sum_{i=1}^{t} v_i x_{ij}^U + \sum_{i=t+1}^{m} v_{i1}\gamma 1_{ij}^U + \sum_{i=t+1}^{m} v_{i2}\gamma 2_{ij}^U\\
&U(Y_j^L) = \sum_{r=1}^{d} u_r y_{rj}^L + \sum_{r=d+1}^{s} u_{r1}\delta 1_{rj}^L + \sum_{r=d+1}^{s} u_{r2}\delta 2_{rj}^L\\
&U(Y_j^U) = \sum_{r=1}^{d} u_r y_{rj}^U + \sum_{r=d+1}^{s} u_{r1}\delta 1_{rj}^U + \sum_{r=d+1}^{s} u_{r2}\delta 2_{rj}^U\\
&u_{r1} - c_r u_{r2} \ge 0, \; r = d+1,\dots,s \quad (c_r > 1)\\
&-v_{i1} + v_{i2} z_i \ge 0 \;, \; i = t+1,\dots,m \quad (0 < z_i < 1)\\
&u_r, v_i \ge 0 \quad r = 1,\dots,d, \; i = 1,\dots t\\
&u_{r1}, u_{r2}, v_{i1}, v_{i2} \ge 0 \quad r = d+1,\dots,s, \; i = t+1,\dots,m
\end{aligned} \tag{14}$$

Lower efficiency bound $h_{j_0}^L$

$$\begin{aligned}
&\max h_{j_0}^L = U(Y_{j_0}^L)\\
&s.t.\\
&U(X_{j_0}^U) = 1\\
&U(Y_{j_0}^L) - U(X_{j_0}^U) \le 0\\
&U(Y_j^U) - U(X_j^L) \le 0 \quad j = 1,\dots,n, \quad j \ne j_0\\
&U(X_j^L) = \sum_{i=1}^{t} v_i x_{ij}^L + \sum_{i=t+1}^{m} v_{i1}\gamma 1_{ij}^L + \sum_{i=t+1}^{m} v_{i2}\gamma 2_{ij}^L\\
&U(X_j^U) = \sum_{i=1}^{t} v_i x_{ij}^U + \sum_{i=t+1}^{m} v_{i1}\gamma 1_{ij}^U + \sum_{i=t+1}^{m} v_{i2}\gamma 2_{ij}^U\\
&U(Y_j^L) = \sum_{r=1}^{d} u_r y_{rj}^L + \sum_{r=d+1}^{s} u_{r1}\delta 1_{rj}^L + \sum_{r=d+1}^{s} u_{r2}\delta 2_{rj}^L\\
&U(Y_j^U) = \sum_{r=1}^{d} u_r y_{rj}^U + \sum_{r=d+1}^{s} u_{r1}\delta 1_{rj}^U + \sum_{r=d+1}^{s} u_{r2}\delta 2_{rj}^U\\
&u_{r1} - c_r u_{r2} \ge 0, \; r = d+1,\dots,s \quad (c_r > 1)\\
&-v_{i1} + v_{i2} z_i \ge 0 \;, \; i = t+1,\dots,m \quad (0 < z_i < 1)\\
&u_r, v_i \ge 0 \quad r = 1,\dots,d, \; i = 1,\dots t\\
&u_{r1}, u_{r2}, v_{i1}, v_{i2} \ge 0 \quad r = d+1,\dots,s, \; i = t+1,\dots,m
\end{aligned} \tag{15}$$

The last two constraints in the linear programs (14) and (15) drive the concavity and the convexity of the value functions for the non-linear outputs and inputs respectively. The parameters c_r and v_i are case-depended user-defined constants that adjust the sharpness of diminishing (increasing) returns. The higher is the value of the parameter c_r the sharper is the effect of the diminishing returns in the efficiency assessments. Analogously, the smaller is the value of v_i the more intense is the increasing returns, which in the case of inputs interprets as sharper penalization of the excess inputs, i.e. input values exceeding the threshold $x_i^o \in [x_i^{\min}, x_i^{\max}]$.

ILLUSTRATIVE EXAMPLE

To illustrate our approach, consider the following example with 21 units, which are evaluated in terms of two inputs x_1, x_2 and two outputs y_1, y_2, all of interval type (see Table 2).

We assume that the output y_2 is non-linear, exhibiting diminishing returns beyond the threshold value y_2^0=63. Thus the virtual output for y_2 is modeled as a piece-wise linear non-decreasing concave value function. Applying the data transformations described in (10), we get the augmented interval data set of Table 3. In accordance to Table 1 and with reference to the non-linear output y_2, the units can be partitioned in three classes, with respect to the position of their interval output relative to the threshold value y_2^0=63: The units s1-s7 belong in class (i), the units s8-s14 belong in class (ii), whereas the units s15-s21 belong in class (iii) (see Figure 2).

Models (14) and (15) applied to the interval data set of Table 3 enable the estimation of the upper and lower efficiency bounds in the presence of the non-linear output y_2. To control the sharpness of concavity of the non-linear output y_2 we set the parameter value in models (14) and (15) at c_2=3, i.e. we introduce the constraint $u_{21} - 3u_{22} \ge 0$ for the unique non-linear output y_2. The higher the value of the parameter c_2 the more sharply the diminishing

Table 2. Input/output data of interval type

Unit	x_1	x_2	y_1	y_2
s1	[118 125]	[74 79]	[209 213]	[56 59]
s2	[101 113]	[37 42]	[208 218]	[57 59]
s3	[115 121]	[23 36]	[259 265]	[51 57]
s4	[111 120]	[69 75]	[216 222]	[58 61]
s5	[108 141]	[42 47]	[222 239]	[60 62]
s6	[109 149]	[25 45]	[241 252]	[58 61]
s7	[105 151]	[35 40]	[250 260]	[45 58]
s8	[130 135]	[87 88]	[205 211]	[56 64]
s9	[121 136]	[75 79]	[206 208]	[52 65]
s10	[125 152]	[59 69]	[201 208]	[58 71]
s11	[121 138]	[65 87]	[267 297]	[60 71]
s12	[125 165]	[28 36]	[260 261]	[52 69]
s13	[138 162]	[30 52]	[248 265]	[58 68]
s14	[143 151]	[49 52]	[211 231]	[49 71]
s15	[139 155]	[35 38]	[215 219]	[65 69]
s16	[171 183]	[67 78]	[209 215]	[69 75]
s17	[169 153]	[55 67]	[204 208]	[67 71]
s18	[139 159]	[28 43]	[241 246]	[68 70]
s19	[163 178]	[52 78]	[271 281]	[67 74]
s20	[159 163]	[60 70]	[262 272]	[65 74]
s21	[176 193]	[27 47]	[228 246]	[71 75]

returns affects the contribution of the output y_2 to the efficiency score of the evaluated unit. Table 4 presents the upper and lower efficiency scores as estimated first by the interval DEA models (8) and (9), where all inputs and outputs are assumed linear and then by the new models (14), (15) that assume a non-linear value function for the interval output y_2.

It is made clear from the results in Table 4 that the assumption of diminishing returns for output y_2 has as effect a general decrease in the efficiency scores of the units. The effect is more intense for the units in class (iii), whose output values for y_2 exceed the threshold value $y_2^0 = 63$.

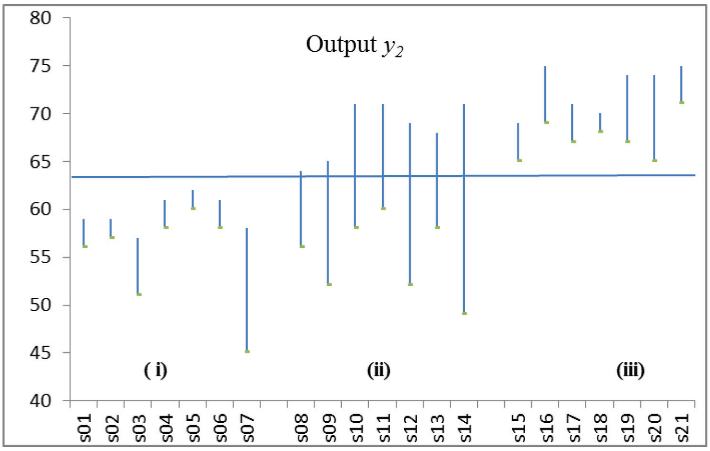
Table 3. The augmented interval data set

Units	x_1	x_2	y_1	$[\delta 1^L_{2j}, \delta 1^U_{2j}]$	$[\delta 2^L_{2j}, \delta 2^U_{2j}]$
s1	[118 125]	[74 79]	[209 213]	[56 59]	[0 0]
s2	[101 113]	[37 42]	[208 218]	[57 59]	[0 0]
s3	[115 121]	[23 36]	[259 265]	[51 57]	[0 0]
s4	[111 120]	[69 75]	[216 222]	[58 61]	[0 0]
s5	[108 141]	[42 47]	[222 239]	[60 62]	[0 0]
s6	[109 149]	[25 45]	[241 252]	[58 61]	[0 0]
s7	[105 151]	[35 40]	[250 260]	[45 58]	[0 0]
s8	[130 135]	[87 88]	[205 211]	[56 63]	[0 1]
s9	[121 136]	[75 79]	[206 208]	[52 63]	[0 2]
s10	[125 152]	[59 69]	[201 208]	[58 63]	[0 8]
s11	[121 138]	[65 87]	[267 297]	[60 63]	[0 8]
s12	[125 165]	[28 36]	[260 261]	[52 63]	[0 6]
s13	[138 162]	[30 52]	[248 265]	[58 63]	[0 5]
s14	[143 151]	[49 52]	[211 231]	[49 63]	[0 8]
s15	[139 155]	[35 38]	[215 219]	[63 63]	[2 6]
s16	[171 183]	[67 78]	[209 215]	[63 63]	[6 12]
s17	[169 153]	[55 67]	[204 208]	[63 63]	[4 8]
s18	[139 159]	[28 43]	[241 246]	[63 63]	[5 7]
s19	[163 178]	[52 78]	[271 281]	[63 63]	[4 11]
s20	[159 163]	[60 70]	[262 272]	[63 63]	[2 11]
s21	[176 193]	[27 47]	[228 246]	[63 63]	[8 12]

Table 5 exhibits the sample linear program (14), which estimates the upper bound of efficiency scores h^U_{14} for the unit s14, which was initially rated efficient by model (8) and then inefficient by the model (14).

The optimal weights u_{21} and u_{22} estimated by the linear program of Table 5 for the two sub-intervals [45, 63] and [63, 75] of the non-linear output y_2 of unit s14 are u_{21}=0.0141 and u_{22}=0.0047. Figure 3 depicts the associated value function assessed for the non-linear output y2 (solid line) together with the linear value function of y_2, as assessed by model (8) for the same unit s14 (dotted line).

Figure 2. The interval data of output y_2



CONCLUSION

This paper extends the piece-wise linear approach, originally developed for standard DEA models, to the case of interval DEA. Our modeling approach concludes with a couple of linear programs capable of estimating a bounded efficiency interval $[h_j^L, h_j^U]$ for each evaluated unit in the presence of input/output dimensions, for which the linearity assumption for the value functions permeating the standard DEA methodology cannot be validated. The models (14) and (15) assume concave value functions for non-linear outputs and convex value functions for the non-linear inputs. However, as long as the form of the value functions is specified through constraints on the weights, any other case-specific form can be defined by

Figure 3. Partial value functions as assessed for output y_2 of unit s14

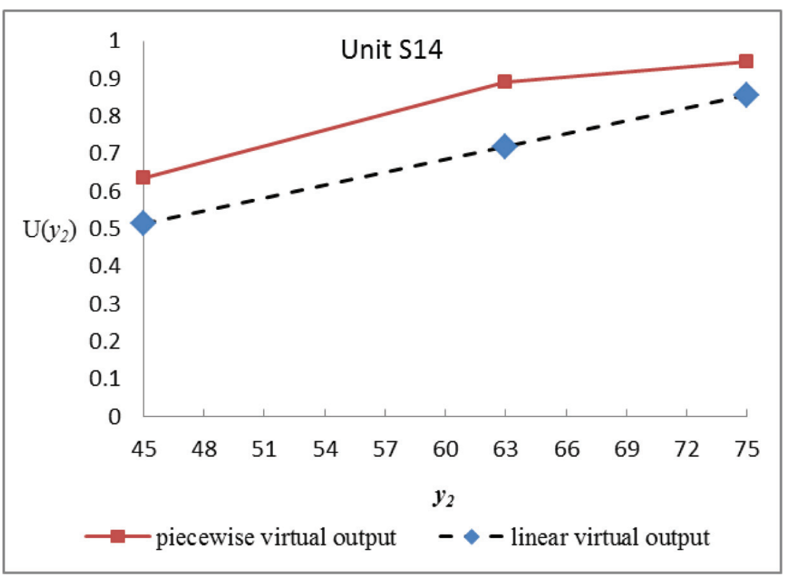


Table 4. Lower and upper bounds of efficiency scores

	Models (8) and (9)		Models (14) and (15) with c_2=3	
Units	h^L	h^U	h^L	h^U
s01	0.7631	0.9909	0.7660	0.9912
s02	0.8809	1	0.8797	1
s03	0.8845	1	0.8821	1
s04	0.8233	1	0.8265	1
s05	0.7358	1	0.7353	1
s06	0.6998	1	0.7115	1
s07	0.7094	1	0.7065	1
Average of Class (i)	0.7852	0.9987	0.7868	0.9987
s08	0.7065	0.9753	0.7101	0.9658
s09	0.6514	1	0.6652	1
s10	0.6511	1	0.6510	1
s11	0.785	1	0.7821	1
s12	0.6768	1	0.6756	1
s13	0.6378	1	0.6401	1
s14	0.5773	1	0.5802	0.9508
Average of Class (ii)	0.6694	0.9964	0.6720	0.9880
s15	0.7452	1	0.7302	1
s16	0.6438	0.8688	0.6081	0.7765
s17	0.7476	0.8871	0.7198	0.8263
s18	0.7539	1	0.7173	1
s19	0.6474	0.9762	0.6420	0.9034
s20	0.6864	0.9260	0.6891	0.8805
s21	0.6537	1	0.6052	1
Average of Class (iii)	0.6968	0.9511	0.6730	0.91239

Table 5. The LP program (14) for the unit s14: estimation of h_{14}^{U}

v_1	v_2	u_1	u_{21}	u_{22}		**RHS**
143	49	0	0	0	=	1
125	79	209	56	0	≤	0
113	42	208	57	0	≤	0
121	36	259	51	0	≤	0
120	75	216	58	0	≤	0
141	47	222	60	0	≤	0
149	45	241	58	0	≤	0
151	40	250	45	0	≤	0
135	88	205	56	0	≤	0
136	79	206	52	0	≤	0
152	69	201	58	0	≤	0
138	87	267	60	0	≤	0
165	36	260	52	0	≤	0
162	52	248	58	0	≤	0
143	49	231	63	8	≤	0
155	38	215	63	2	≤	0
183	78	209	63	6	≤	0
153	67	204	63	4	≤	0
159	43	241	63	5	≤	0
178	78	271	63	4	≤	0
163	70	262	63	2	≤	0
193	47	228	63	8	≤	0
0	0	0	-1	3	≤	0
0	0	231	63	8	max	h_{14}^{U}

restricting the weights appropriately. Our modeling approach enables models (14) and (15) to carry out the efficiency assessments when different types of input/output measures are involved, such as linear and/or non-linear inputs and outputs with exact as well as with interval measures.

ACKNOWLEDGMENT

This research has been co-financed by the European Union (European Social Fund – ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: THALES-Investing in knowledge society through the European Social Fund.

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