A novel DEA approach to assess individual and overall efficiencies in two-stage processes

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Abstract

In the DEA context, a two-stage production process assumes that the first stage transforms external inputs to a number of intermediate measures, which then are used as inputs to the second stage that produces the final outputs. In the additive approach, the overall efficiency of the production process is defined as a weighted average of the efficiencies of the individual stages. As the weights are assumed functions of the DEA multipliers, they derive endogenously by the optimization process and are different for each evaluated unit. In this paper, we first show that the above assumption made for the weights unduly bias the efficiency assessments in favor of the second stage and we present an unbiased approach to assess the efficiencies of the two stages in an additive two-stage DEA framework. Then, we use the envelopment variants of the individual stages as models to develop a two-phase procedure, which enables the derivation of the efficient frontier at a minimum distortion of the intermediate measures.

Keywords: *Data envelopment analysis (DEA); Two-stage DEA; Projections; Efficient frontier.*

Introduction

Data Envelopment Analysis (DEA) (Charnes et al.,1978) is a widely used technique for evaluating the performance of peer decision making units (DMUs) that consume multiple inputs to produce multiple outputs. In conventional DEA a single stage production process is assumed, which transforms inputs to final outputs, ignoring the internal structure of the decision making units. However, there is an increasing literature body that is devoted to the efficiency assessment in multistage production processes. Castelli et al. (2010) provide a comprehensive categorized overview of models and methods developed for different multi-stage production architectures. In this paper, we focus on the typical architecture of a two-stage production process, which assumes that the external inputs entering the first stage of the process are transformed to a number of intermediate measures which then are used as inputs to the second stage that produces the final outputs.

Seiford and Zhu (1999) studied such a production process in the banking sector by treating the two stages independently, i.e. without assuming any relationship

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between the two stages. Kao and Hwang (2008) introduced the multiplicative approach that takes into account a series relationship of the two stages. They developed a model that decomposes the overall efficiency to the product of the efficiencies of the two stages. Their approach is based on the reasonable assumption that the virtual intermediate measures are the same, no matter if they are considered as outputs of the first stage or inputs to the second stage. Chen et al. (2009) introduced the additive efficiency decomposition. They decompose the overall efficiency of the production process to a weighted average of the efficiencies of the individual stages. Their modeling approach assumes that the weighting of the two stages derives endogenously by the optimization process. Notably however, that the assumption made for the weights unduly bias the efficiency assessments in favor of the second stage.

Our paper focuses on an alternative approach in two-stage DEA under the common assumption of the series relationship of the two stages. The selection of different orientations for the two stages enables us to aggregate the efficiency measures of the two individual stages in a bi-objective linear programming framework. Our model estimates simultaneously optimal efficiency scores for the two stages, which then are used to calculate the overall efficiency of the production process, by selecting the aggregation method a posteriori. Thus our method, as opposed to the aforementioned decomposition approaches, is considered as a composition approach. Then we present a two-phase procedure to derive the efficient frontier at a minimum distortion of the intermediate measures.

The additive decomposition approach: a critical review

Assume n DMUs ($j=1,...,n$), each using m external inputs $X_j=(x_{ij}, i=1,...,m)$, in the first stage to produce q outputs $Z_j=(z_{pj}, p=1,...,q)$ from that stage. The outputs obtained from the first stage are then used as inputs to the second stage to produce s final outputs $Y_j=(y_{rj}, r=1,...,s)$. The vectors $v=(v_1,...,v_m)$, $w=(w_1,...,w_q)$ and $u=(u_1,...,u_s)$ are variable weights associated with the external inputs, the intermediate measures and the final outputs, respectively whereas e_j^o, e_j^1, e_j^2 denote the overall, the first stage and the second stage efficiency of unit j.

Chen et al. (2009) introduced the additive decomposition approach; its major characteristic is that apart from the definition of the efficiency of the two individual stages, it premises the definition of the overall efficiency (1) together with a model to decompose the overall efficiency to the stage efficiencies (2). Then, the efficiency scores of the two stages, for each unit, derive as offspring of the overall efficiency.

$$e_j^o=\frac{uY_j+wZ_j}{vX_j+wZ_j}, \quad e_j^1=\frac{wZ_j}{vX_j}, \quad e_j^2=\frac{uY_j}{wZ_j} \qquad (1)$$

$$\frac{uY_j+wZ_j}{vX_j+wZ_j}=t_j^1\frac{wZ_j}{vX_j}+t_j^2\frac{uY_j}{wZ_j}, \quad t_j^1+t_j^2=1 \qquad (2)$$

$$t_j^1=\frac{vX_j}{vX_j+wZ_j}, \quad t_j^2=\frac{wZ_j}{vX_j+wZ_j} \qquad (3)$$

In effect, the overall efficiency is *a weighted arithmetic average* of the stage efficiencies. The weights (3) are obtained by solving the system (2) for t_j^1 and t_j^2. Particularly, as the weights are functions of the virtual measures, they depend on the unit being evaluated and, obviously, they generally differentiate from one unit to another. On the basis of the above definitions, the linear model (4) has been proposed to assess the overall efficiency of the evaluated unit j_0. Once an optimal solution (v^*, w^*, u^*) of model (4) is obtained, the overall efficiency and the stage efficiencies are calculated by the following equations (5).

$$\begin{aligned} e_{j_0}^o &= \max uY_{j_0} + wZ_{j_0} \\ s.t.& \\ vX_{j_0} + wZ_{j_0} &= 1 \\ uY_j - wZ_j &\le 0,\ j = 1,\dots,n \\ wZ_j - vX_j &\le 0,\ j = 1,\dots,n \\ v \ge 0, w &\ge 0, u \ge 0 \end{aligned} \qquad (4)$$

$$\begin{aligned} e_{j_o}^o &= u^* Y_{j_0} + w^* Z_{j_0} \\ t_{j_0}^1 &= v^* X_{j_0}, \quad t_{j_0}^2 = w^* Z_{j_0} \\ e_{j_o}^1 &= \frac{w^* Z_{j_0}}{v^* X_{j_0}} = \frac{t_{j_0}^2}{t_{j_0}^1} \\ e_{j_o}^2 &= \frac{e_{j_o}^o - t_{j_0}^1 e_{j_o}^1}{t_{j_0}^2} = \frac{u^* Y_{j_0}}{w^* Z_{j_0}} \end{aligned} \qquad (5)$$

The overall efficiency $e_{j_0}^o$ is obtained as the optimal value of the objective function in (4), the weight $t_{j_0}^1$ is obtained as the optimal virtual input, the weight $t_{j_0}^2$ is obtained as the optimal virtual intermediate measure, the efficiency of the first stage $e_{j_0}^1$ is given by the ratio of the two weights, whereas the efficiency of the second stage $e_{j_0}^2$ is obtained as offspring of $e_{j_0}^o, e_{j_0}^1$. In Chen et al. (2009), the definition of the overall efficiency, as in (1), is implicit. Explicit is, however, the definition of the weights (3), which is made for the sake of linearization of the efficiency assessment model, in the form of (4). The argument given for the weights is that they represent the relative contribution of the two stages to the overall performance of the DMU. The "size" of each stage, as measured by the portion of total resources devoted to each stage, is assumed to reflect their relative contribution to the overall efficiency of the DMU. However, as long as the weights derive from the optimal solution of (4), they depend on the DMU being evaluated and, generally, they are different for different DMUs. Thus, the "size" of a stage is not an objective reality, as it is viewed differently from each DMU. But this is not the only peculiarity emerging from the definition of the weights. Indeed, from the definition of the weights (3), as well as form (5) holds that $\frac{t_j^2}{t_j^1} = \frac{wZ_j}{vX_j} = e_j^1 \le 1$ i.e. $t_j^2 \le t_j^1$. Thus, the efficiency decomposition in model (4) is biased in favor of the second stage, as the efficiency assessments are always made by imposing to the first stage a greater or equal weight than the weight assigned to the second stage. This is a major drawback of the additive decomposition method.

The composition approach: A reverse perspective

Consider the output-oriented CRS model (6) for the first stage and the input-oriented CRS model (7) for the second stage, where the same intermediate weights are assumed for both stages. Appending the constraints $uY_j - wZ_j \le 0, j = 1,\dots,n$ to model (6) and the constraints $wZ_j - vX_j \le 0, j = 1,\dots,n$ to model (7) we derive two augmented models for the first and the second stage respectively.

1st Stage – Output Oriented

$$\frac{1}{e_{j_0}^1} = \min vX_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1 \qquad (6)$$
$$wZ_j - vX_j \le 0,\ j = 1,\dots,n$$
$$v \ge 0, w \ge 0$$

2nd Stage – Input Oriented

$$\max uY_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1 \qquad (7)$$
$$uY_j - wZ_j \le 0,\ j = 1,\dots,n$$
$$w \ge 0, u \ge 0$$

Single-Objective Linear Program

$$\min vX_{j_0} - uY_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1 \qquad (8)$$
$$uY_j - wZ_j \le 0,\ j = 1,\dots,n$$
$$wZ_j - vX_j \le 0,\ j = 1,\dots,n$$
$$v \ge 0, w \ge 0, u \ge 0$$

Notice that an optimal solution of model (6) is also optimal in the augmented model. Analogously, an optimal solution of model (7) is also optimal in the augmented variant. The augmented models have common constraints and, thus, can be jointly consider in a bi-objective linear program. The basic model (8) derives by aggregating the two objective functions additively. Once an optimal solution (u^*, v^*, w^*) of model (8) is obtained, the efficiency scores for unit j_0 in the first and the second stage are respectively $\hat{e}_{j_0}^1 = 1/v^* X_{j_0}$ and $\hat{e}_{j_0}^2 = u^* Y_{j_0}$. Having the stage efficiency scores, the overall efficiency can be computed either as a simple or as a weighted average, with the weights given a priori commonly for all the units. Hence, our approach can be considered as "neutral", as opposed to the Chen's et al. (2009) one, where the unit under evaluation assigns its own weights to the efficiency scores of the two individual stages. The essential characteristic of our method is that the overall efficiency is derived from the stage efficiencies (composition approach), whereas, under the decomposition frameworks (Chen et al, 2009 and Kao and Hwang, 2008) the stage efficiencies derive from the overall efficiency.

Comparing the efficiency scores derived from model (8) with those obtained by Chen et al. (2009) method, on randomly generated data as well as on data sets reported in the literature, showed that they differ significantly. It is conceivable that the overall efficiency scores cannot be compared directly. Moreover, as post-optimality analysis verified, model (8) yields unique efficiency scores.

Deriving the efficient frontier

An issue worth mentioning is the inability of two-stage DEA models to derive the efficient frontier, i.e. to provide sufficient information on how to project the inefficient units on the DEA frontier. As mentioned by Chen et al. (2010) the standard DEA technique of adjusting the inputs and outputs by the efficiency scores cannot yield a frontier projection under the concept of a two-stage process neither in their additive approach nor in Kao and Hwang's (2008) multiplicative approach. They addressed this issue by developing alternative models, under the framework of Kao and Hwang (2008), which generated a set of new inputs, outputs and intermediate measures that constituted efficient projections. Unfortunately, the aforementioned technique cannot be applied in the additive framework. Recently, Chen et al. (2013) noticed that the envelopment and the multiplier forms are two types of network DEA models, which use different concepts of efficiency. Consequently, the envelopment forms of network DEA models should be used for determining the frontier projection for inefficient DMUs while the multiplier models for estimating efficiency scores.

In our approach, the envelopment (dual) form of our basic model (8) does not provide adequate information to project the units onto the efficient frontier. To address the deficiencies discussed above, we apply a reverse technique by selecting an input orientation for the first stage (9) and an output orientation for the second stage (10). Appending the constraints of model (9) to model (10), and vice versa, we derive two augmented models for the first and the second stage respectively which have common constraints; hence they enable us to jointly consider them as a bi-objective linear program. Accordingly, by aggregating the two objective functions additively, we derive the following single-objective linear program (11).

1st Stage – Input Oriented	2nd Stage – Output Oriented	Phase I	Phase II
$$\begin{aligned} &\min \theta_1 \\ &s.t. \\ &X\lambda \le \theta_1 X_0 \quad (9) \\ &Z\lambda \ge Z_0 \\ &\lambda \ge 0 \end{aligned}$$	$$\begin{aligned} &\max \theta_2 \\ &s.t. \\ &Y\mu \ge \theta_2 Y_0 \quad (10) \\ &Z\mu \le Z_0 \\ &\mu \ge 0 \end{aligned}$$	$$\begin{aligned} &\min \theta_1 - \theta_2 \\ &s.t. \\ &X\lambda \le \theta_1 X_0 \\ &Y\mu \ge \theta_2 Y_0 \\ &Z\lambda \ge Z_0 \quad (11) \\ &Z\mu \le Z_0 \\ &\lambda \ge 0, \mu \ge 0 \\ &\theta_1 \le 1, \theta_2 \ge 1 \end{aligned}$$	$$\begin{aligned} &\max M(es^- + es^+) - (e\alpha + e\beta) \\ &s.t. \\ &X\lambda + s^- = \theta_1^* X_0 \\ &Y\mu - s^+ = \theta_2^* Y_0 \\ &Z\lambda + \alpha - \beta \ge Z_0 \quad (12) \\ &Z\mu + \alpha - \beta \le Z_0 \\ &\lambda \ge 0, \mu \ge 0 \\ &s^+ \ge 0, s^- \ge 0 \\ &\alpha \ge 0, \beta \ge 0 \end{aligned}$$

Notice that model (11) in Phase I yields the independent efficiency scores θ_1^*, θ_2^* for the two stages, which then they are passed in Phase II. Once optimal λ*'s and μ*'s are obtained from the Phase II model (12), the efficient projections for the external inputs and the final outputs derive as $\hat{X}_0 = \sum_{j \in J} X_j \lambda_j^*$, $\hat{Y}_0 = \sum_{j \in J} Y_j \mu_j^*$, with adjusted intermediate measures $\hat{Z}_0 = Z_0 - \alpha^* + \beta^*$, where α^*, β^* are the vectors of the optimal values of the deviation variables in (12). The vectors of deviation variables α and β are used to yield new intermediate measures at a minimum distortion of the original ones, while M is a large positive number that gives priority in defining the max-slack solution with respect to the external inputs and the final outputs. The rationale is that as the intermediate measures are debatable between the two stages, they should undergo minor changes from their initial state. Such an issue, is not taken into account in other projection methods (Chen et al. 2010, 2013), where the new estimated intermediate measures differ substantially from their original values and depend on the orientation assumed. The experiments show that the projections render the units efficient.

Conclusions

We showed in this paper that the additive decomposition approach introduced by Chen et al (2009) biases the efficiency assessments in favor of the second stage. Then we introduce the composition approach to two-stage DEA. In principle, we differentiate from the additive decomposition approach in that we estimate first optimal and unbiased efficiency scores for the two stages, which are then aggregated additively as a simple or a weighted average to obtain the overall efficiency. Further, having obtained the individual efficiency scores, one might consider different aggregation schemes. Moreover, we introduce a two-phase procedure to derive the efficient frontier. The novelty of this two-phase approach is that, when projecting the inefficient units on the frontier, the adjusted intermediate measures are as close as possible to their original values, an issue that is not taken into consideration in other approaches. Recapping, the proposed composition approach provides insight where the conventional DEA models do not fully access, by yielding neutral and unique stage efficiencies (the true efficiency scores) and the efficient frontier.

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