Models for unbiased efficiency assessments in an additive two-stage DEA framework

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Ευρωπαϊκή Ένωση
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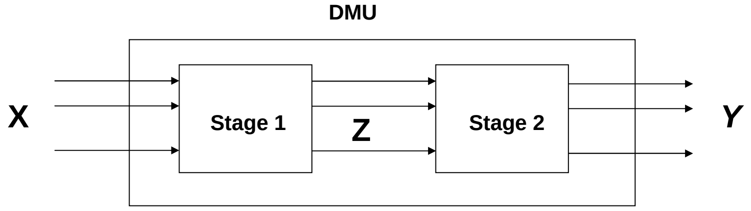
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

Outline

- The basic two-stage process
- The decomposition approach
  - The multiplicative method
  - The additive method
- The composition approach
  - An alternative efficiency assessment method in two-stage DEA
- Comparisons
- Summary
- Deriving the efficient frontier

The basic two-stage process



e_j^o The overall efficiency of the DMU (unit j)

e_j^1 The efficiency of stage-1 (unit j)

e_j^2 The efficiency of stage-2 (unit j)

The decomposition approach

- Definition of the overall efficiency of the DMU
- Definition of the stage efficiencies
- A model to decompose the overall efficiency to the stage efficiencies

e_j^0 first, then e_j^1 and e_j^2

The multiplicative method (Kao & Hwang, 2008)

The square geometric average

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Definitions

$$e_j^o = \frac{uY_j}{vX_j} \qquad e_j^1 = \frac{wZ_j}{vX_j} \qquad e_j^2 = \frac{uY_j}{wZ_j}$$

Decomposition model

$$e_j^o = \frac{uY_j}{vX_j} \cdot \frac{wZ_j}{wZ_j} = \frac{wZ_j}{vX_j} \cdot \frac{uY_j}{wZ_j} = e_j^1 \cdot e_j^2$$

The multiplicative method (Kao & Hwang, 2008)

The square geometric average

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Assessment model

$$e_{j_0}^o = \max uY_{j_0}$$

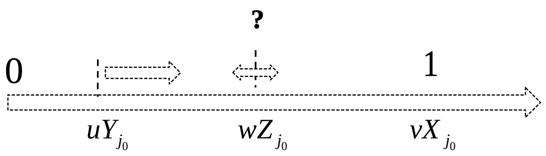
$s.t.$

$vX_{j_0} = 1$

$uY_j - wZ_j \le 0, \; j = 1, \ldots, n$

$wZ_j - vX_j \le 0, \; j = 1, \ldots, n$

$v \ge 0, w \ge 0, u \ge 0$



$$e_{j_o}^o = u^* Y_{j_0}, \quad e_{j_o}^1 = w^* Z_{j_0}, \quad e_{j_o}^2 = \frac{e_{j_o}^o}{e_{j_o}^1}$$

Decomposition is not unique

The additive method (Chen, Cook, Li & Zhu 2009)

The weighted arithmetic average

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Definitions

$$e_j^o = \frac{uY_j + wZ_j}{vX_j + wZ_j} \qquad e_j^1 = \frac{wZ_j}{vX_j} \qquad e_j^2 = \frac{uY_j}{wZ_j}$$

Decomposition model

$$e_j^o = \frac{uY_j + wZ_j}{vX_j + wZ_j} = t_j^1 \frac{wZ_j}{vX_j} + t_j^2 \frac{uY_j}{wZ_j}$$

$$t_j^1 + t_j^2 = 1$$

The additive method (Chen, Cook, Li & Zhu 2009)

The weighted arithmetic average

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The weights derive by solving the following system for t^1 and t^2

$$\frac{uY_j + wZ_j}{vX_j + wZ_j} = t_j^1 \frac{wZ_j}{vX_j} + t_j^2 \frac{uY_j}{wZ_j}$$

$$t_j^1 + t_j^2 = 1$$

$$t_j^1 = \frac{vX_j}{vX_j + wZ_j}, \quad t_j^2 = \frac{wZ_j}{vX_j + wZ_j}$$

The additive method (Chen, Cook, Li & Zhu 2009)

The weighted arithmetic average

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Assessment model

$$
\begin{aligned}
e^o_{j_0} &= \max\, uY_{j_0} + wZ_{j_0} \\
s.t. & \\
vX_{j_0} + wZ_{j_0} &= 1 \\
uY_j - wZ_j &\le 0,\ j = 1,\dots,n \\
wZ_j - vX_j &\le 0,\ j = 1,\dots,n \\
v \ge 0, w &\ge 0, u \ge 0
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
e^o_{j_o} &= u^* Y_{j_0} + w^* Z_{j_0} \\
t^1_{j_0} &= v^* X_{j_0}, \quad t^2_{j_0} = w^* Z_{j_0} \\
e^1_{j_o} &= \frac{w^* Z_{j_0}}{v^* X_{j_0}} = \frac{t^2_{j_0}}{t^1_{j_0}} \\
e^2_{j_o} &= \frac{e^o_{j_o} - t^1_{j_0} e^1_{j_o}}{t^2_{j_0}} = \frac{u^* Y_{j_0}}{w^* Z_{j_0}}
\end{aligned}
$$

Decomposition is unique

The additive method (Chen, Cook, Li & Zhu 2009)

The weighted arithmetic average

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Observe that, by the definition of the weights ...

$$
t^1_j = \frac{vX_j}{vX_j + wZ_j}, \quad t^2_j = \frac{wZ_j}{vX_j + wZ_j}
$$

holds that... $\dfrac{t^2_j}{t^1_j} = \dfrac{wZ_j}{vX_j} = e^1_j \le 1$

Thus ... $t^2_j \le t^1_j$

i.e. is there is a bias favoring the second stage against the first one

The composition approach
(the proposed approach)

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- Assess efficiencies for stage 1 and stage 2, then …
- Aggregate the stage efficiencies to obtain the overall efficiency of the DMU

e_j^1 and e_j^2 first, then e_j^0

- ○ Different aggregation models can be applied

Assessing the stage efficiencies

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Assessment model

$$\begin{aligned}
&\min\ vX_{j_0} - uY_{j_0} \\
&s.t. \\
&wZ_{j_0} = 1 \\
&uY_j - wZ_j \le 0,\ j = 1,\dots,n \\
&wZ_j - vX_j \le 0,\ j = 1,\dots,n \\
&v \ge 0, w \ge 0, u \ge 0
\end{aligned}$$

$$\hat{e}^1_{j_0} = \frac{1}{v^* X_{j_0}} \qquad \hat{e}^2_{j_0} = u^* Y_{j_0}$$

Rationale of the model

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Independent models/augmented models

Stage 1 (Output oriented)

$$\min vX_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1$$
$$wZ_j - vX_j \le 0,\ j = 1,\dots,n$$
$$v \ge 0, w \ge 0$$

$$uY_j - wZ_j \le 0,\ j = 1,\dots,n$$

$$E_{j_o}^1 = \frac{1}{v^* X_{j_0}} \qquad E_{j_o}^2 = u^* Y_{j_0}$$

Stage 2 (Input oriented)

$$\max uY_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1$$
$$uY_j - wZ_j \le 0,\ j = 1,\dots,n$$
$$w \ge 0, u \ge 0$$

$$wZ_j - vX_j \le 0,\ j = 1,\dots,n$$

Independent efficiencies

Rationale of the model

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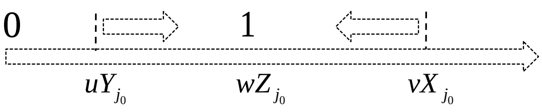
The two LPs have common constraints and, thus, are jointly considered in a bi-objective LP (the assessment model)

$$\min vX_{j_0} - uY_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1$$
$$uY_j - wZ_j \le 0,\ j = 1,\dots,n$$
$$wZ_j - vX_j \le 0,\ j = 1,\dots,n$$
$$v \ge 0, w \ge 0, u \ge 0$$



$$\hat{e}_{j_0}^1 = \frac{1}{v^* X_{j_0}} \qquad \hat{e}_{j_0}^2 = u^* Y_{j_0}$$

The model estimates the stage-1 and stage-2 efficiencies as close as possible to the *zenith* point (1,1,1) by minimizing the $L1$ (Manhattan) norm

Composition of the stage efficiencies to obtain the overall efficiency

Additive aggregation (option 1)

Simple arithmetic average: $\hat{e}^{o}_{j_0} = \frac{1}{2}\hat{e}^{1}_{j_0} + \frac{1}{2}\hat{e}^{2}_{j_0}$

Composition is unique as stage efficiencies are unique

Composition of the stage efficiencies to obtain the overall efficiency

Additive aggregation (option 2)

Weighted arithmetic average: $\hat{e}^{o}_{j_0} = a_1\hat{e}^{1}_{j_0} + a_2\hat{e}^{2}_{j_0}$

$$\begin{aligned}
&\min\ a_1 vX_{j_0} - a_2 uY_{j_0} \\
&s.t. \\
&wZ_{j_0} = 1 \\
&uY_j - wZ_j \le 0,\ j = 1,\dots,n \\
&wZ_j - vX_j \le 0,\ j = 1,\dots,n \\
&v \ge 0, w \ge 0, u \ge 0
\end{aligned}$$

Any α_1, α_2 such that:

$$a_1 + a_2 = 1$$

Composition is unique as stage efficiencies are unique

Composition of the stage efficiencies to obtain the overall efficiency

Additive aggregation (option 3)

Weighted by size:

$$\min\ a_1 vX_{j_0} - a_2 uY_{j_0}$$
$$s.t.$$
$$wZ_{j_0} = 1$$
$$uY_j - wZ_j \le 0,\ j = 1,\dots,n$$
$$wZ_j - vX_j \le 0,\ j = 1,\dots,n$$
$$v \ge 0, w \ge 0, u \ge 0$$

$$a_1 = \frac{e\sum_{j=1}^{n}\bar{X}_j}{e\sum_{j=1}^{n}\bar{X}_j + e\sum_{j=1}^{n}\bar{Z}_j},\quad a_2 = \frac{e\sum_{j=1}^{n}\bar{Z}_j}{e\sum_{j=1}^{n}\bar{X}_j + e\sum_{j=1}^{n}\bar{Z}_j}$$

$$e = (1,1,\dots,1)$$

Composition is unique as stage efficiencies are unique

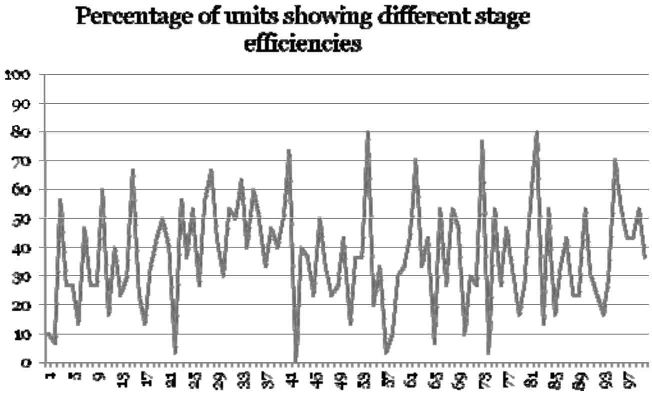
Composition of the stage efficiencies to obtain the overall efficiency

Multiplicative aggregation

$$\hat{e}^{o}_{j_0} = \hat{e}^{1}_{j_0} \cdot \hat{e}^{2}_{j_0} = \frac{1}{v^* X_{j_0}} \cdot u^* Y_{j_0} = \frac{u^* Y_{j_0}}{v^* X_{j_0}}$$

Composition is unique as stage efficiencies are unique

Additive composition Vs additive decomposition



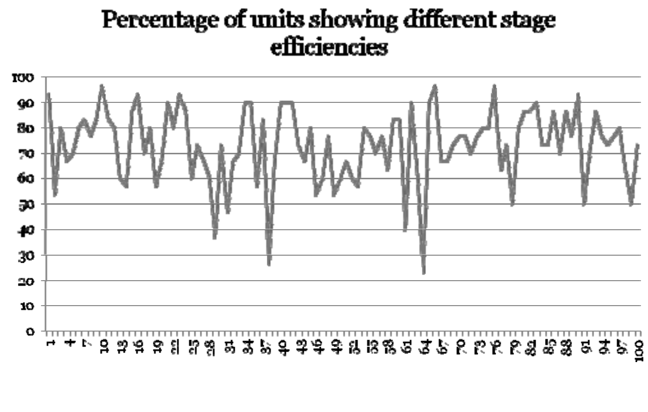
$$\hat{e}^1_{j_0} \geq e^1_{j_0}$$

$$\hat{e}^2_{j_0} \leq e^2_{j_0}$$

↑ favored

$$\hat{e}^0_{j_0} \; ? \; e^0_{j_0}$$

Multiplicative composition Vs multiplicative decomposition



$$\hat{e}^1_{j_0} \geq e^1_{j_0}$$

$$\hat{e}^2_{j_0} \leq e^2_{j_0}$$

$$\hat{e}^0_{j_0} \leq e^0_{j_0}$$

Locating different efficiency scores

$$
\begin{aligned}
&\min\ tvX_{j_0} - (1-t)uY_{j_0} \qquad t \in [0,1] \\
&s.t. \\
&wZ_{j_0} = 1 \\
&uY_j - wZ_j \le 0,\ j = 1,\dots,n \\
&wZ_j - vX_j \le 0,\ j = 1,\dots,n \\
&v \ge 0, w \ge 0, u \ge 0
\end{aligned}
$$

For different values of the parameter t , the model above can locate the efficiency scores derived by the other methods

Locating different efficiency scores: explaining similarities/dissimilarities

DMU	t	t^2	e^1	e^2	ADM	CM	MDM
7	(0, 0.048)		0.300	0.538			
	[0.048, 0.0528)		0.382	0.502			
	[0.0528, 0.0575)		0.514	0.464			
	[0.0575, 0.1368)		0.575	0.452			
	[0.1368, 0.2718)		0.671	0.412			*
	[0.2718, 1)	0.429	0.752	0.352	*	*	
16	(0, 0.0281)		0.599	0.385			
	[0.0281, 0.0504)		0.744	0.375			
	[0.0504, 0.1406)		0.869	0.365			
	[0.1406, 0.491)	0.470	0.886	0.362	*		*
	[0.491, 1)		0.907	0.336		*	

Summary

- The additive decomposition bias the efficiency assessments in favor of the second stage
- The multiplicative decomposition is not unique
- The proposed composition approach:
  - Provides **unbiased** and **unique** stage efficiencies

    Enables considering a posteriori different composition models
  - Can locate the efficiency scores derived by the decomposition methods. The inverse does not hold.

Deriving the efficient frontier: A 2-phase approach

- In standard DEA, one can obtain the efficiency scores and locate the efficient frontier by solving either the multiplier or the envelopment (dual) form of the DEA LPs.
- This does not hold in two-stage DEA
- The duals of the two-stage DEA models do not provide sufficient information to locate correctly the efficient frontier
- The multiplier models should be used for efficiency scores, adjusted envelopment models should be used for the efficient frontier (Chen et al. 2013)

Deriving the efficient frontier: A 2-phase approach

Stage 1: Input oriented

$$
\begin{aligned}
&\min \theta_1 \\
&s.t. \\
&X\lambda \le \theta_1 X_0 \\
&Z\lambda \ge Z_0 \\
&\lambda \ge 0 \\
&Y\mu \ge \theta_2 Y_0 \\
&Z\mu \le Z_0 \\
&\mu \ge 0
\end{aligned}
$$

Stage 2: Output oriented

$$
\begin{aligned}
&\max \theta_2 \\
&s.t. \\
&Y\mu \ge \theta_2 Y_0 \\
&Z\mu \le Z_0 \\
&\mu \ge 0 \\
&X\lambda \le \theta_1 X_0 \\
&Z\lambda \ge Z_0 \\
&\lambda \ge 0
\end{aligned}
$$

Deriving the efficient frontier: A 2-phase approach

The two LPs have common constraints and, thus, are jointly considered in a bi-objective LP (the phase I model)

Phase I

$$
\begin{aligned}
&\min \theta_1 - \theta_2 \\
&s.t. \\
&X\lambda \le \theta_1 X_0 \\
&Y\mu \ge \theta_2 Y_0 \\
&Z\lambda \ge Z_0 \\
&Z\mu \le Z_0 \\
&\lambda \ge 0, \mu \ge 0 \\
&\theta_1 \le 1, \theta_2 \ge 1
\end{aligned}
$$

θ_1^*, θ_2^* ⇒

Phase II

$$
\begin{aligned}
&\max M(es^- + es^+) - (e\alpha + e\beta) \\
&s.t. \\
&X\lambda + s^- = \theta_1^* X_0 \\
&Y\mu - s^+ = \theta_2^* Y_0 \\
&Z\lambda + \alpha - \beta \ge Z_0 \\
&Z\mu + \alpha - \beta \le Z_0 \\
&\lambda \ge 0, \mu \ge 0 \\
&s^+ \ge 0, s^- \ge 0 \\
&\alpha \ge 0, \beta \ge 0
\end{aligned}
$$

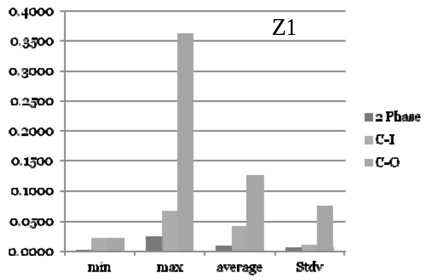
$$\hat{X}_0 = \sum_{j \in J} X_j \lambda_j^*,\ \hat{Y}_0 = \sum_{j \in J} Y_j \mu_j^*$$

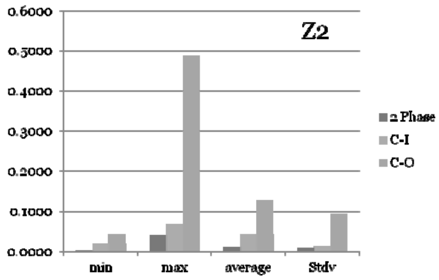
$$\hat{Z}_0 = Z_0 - \alpha^* + \beta^*$$

Intermediate measures: MSD (original, estimated)

100 data sets - 100 DMUS (2,2,2) - normal distribution (0.5,0.1)

	2 Phase		Chen et al (2009)_input oriented		Chen et al (2009, 2012)-output oriented	
	Z1	Z2	Z1	Z2	Z1	Z2
min	0.0008	0.0018	0.0218	0.0201	0.0210	0.0422
max	0.0251	0.0396	0.0672	0.0668	0.3615	0.4873
average	0.0092	0.0105	0.0417	0.0415	0.1263	0.1283
Stdv.	0.0059	0.0076	0.0095	0.0124	0.0751	0.0950





Intermediate measures: MSD (original, estimated)

100 data sets - 100 DMUS (2,3,2) - normal distribution (0.5,0.1)

	2 Phase			Chen et al (2009)_input oriented			Chen et al (2009, 2012)-output oriented		
	Z1	Z2	Z3	Z1	Z2	Z3	Z1	Z2	Z3
min	0.0044	0.0014	0.0044	0.0197	0.0203	0.0210	0.0289	0.0403	0.0343
max	0.0531	0.0325	0.0469	0.0647	0.0646	0.0563	0.5840	0.4903	0.6087
average	0.0141	0.0120	0.0127	0.0344	0.0361	0.0355	0.1087	0.0964	0.1016
Stdv.	0.0082	0.0070	0.0083	0.0096	0.0094	0.0090	0.0876	0.0728	0.0853