Value-based data envelopment analysis: a piece-wise linear programming approach

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Abstract: Incorporating value judgments in data envelopment analysis (DEA) is a broad methodological framework that facilitates driving the efficiency assessments in line with individual preferences. Value-based DEA is a recent development that resorts to value assessment protocols from multiple criteria decision analysis (MCDA) to transform the original input/output data to a value scale. In this context, we introduce in this paper a piece-wise linear programming approach to value-based DEA, by employing a data transformation-variable alteration technique and assurance region constraints. We outline our approach by revisiting a previous efficiency assessment study drawn from the literature.

Keywords: data envelopment analysis; DEA; piece-wise linear programming; value-based DEA.

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1 Introduction

Data envelopment analysis (DEA) is the leading technique for measuring the relative efficiency of decision making units (DMUs) that use multiple inputs to produce multiple outputs. The two milestone DEA models, namely the CCR (Charnes et el., 1978) and the BCC (Banker et al., 1984) models have become standards in the literature of performance measurement, under the assumption of constant (CRS) and variable (VRS) returns to scale respectively. Both are stated and solved either in the multiplier forms or their duals, the envelopment forms. In terms of the multiplier form, the efficiency of a DMU is explicitly defined on a bounded scale by the ratio of a weighted sum of its outputs to a weighted sum of its inputs. The weights are estimated in favour of the evaluated unit, so as to maximise its relative efficiency. The envelopment form, along with the efficiency scores provides the projections of the inefficient units on the efficient frontier, by assuming either an input or an output orientation. Another frequently used model is the additive DEA model (Charnes et al., 1985). The basic additive model does not assume any prescribed orientation when it projects the inefficient units on the efficient frontier. A limitation of this model is that it only discriminates the set of DMUs in efficient and inefficient ones, with its efficiency measure lacking of an intuitive interpretation. A variant of the additive model is the weighted additive model (Ali et al., 1995).

Although the flexibility privileged to the evaluated unit in selecting its own weights is one of the major advantages of DEA in locating inefficiencies, the weights assigned to the inputs and outputs may not be necessarily in line with the individual preferences of a decision maker in his efficiency assessment project. To address this issue, various methods to incorporate value judgments in DEA efficiency assessments have been arisen. The necessity to drive the weights assigned to the factors originates from a variety of reasons, such as to improve the discrimination power of DEA, to restrain the diversity of the weights assigned to the same factor by different DMUs and to incorporate individual preferences and trade-offs over the inputs and outputs.

There are two broad classes of methods to incorporate value judgments in DEA. The one is based on an explicit restriction of the weight space by imposing either direct constraints on the weight variables (Cook et al., 1991; Dyson and Thanassoulis, 1988; Thompson et al., 1986, 1990) or constraints on the virtual inputs and outputs, i.e. the input/output measures multiplied by the weights (Wong and Beasley, 1990). The other concerns the alteration of the data space, either by altering the dataset itself, such as the cone-ratio approach of Charnes et al. (1989), or by introducing fictitious DMUs (Golany and Roll, 1994; Halme et al., 1999; Podinovski, 2004; Thanassoulis and Allen, 1998). The reader is referred to Thanassoulis et al. (2004) for a comprehensive review and interpretations of the various methods. Gouveia et al. (2008) and recently Almeida and Dias (2012) presented a hybrid approach (they successfully named it ‘value-based DEA’) originated from multiple criteria decision analysis (MCDA). Their approach transforms the original input/output data from their original scales to a value scale, on the basis of preference information delivered by the decision maker. They use the weighted additive DEA model for the efficiency assessments. Imposing restrictions on the weights, they drive the projections of the inefficient units on the efficient frontier according to the user’s preferences.

Motivated by Almeida and Dias (2012), we introduce in this paper an alternative value-based DEA approach that embeds explicit formulations of the value functions assumed for the decision maker in a DEA modelling framework. To this end, we rely on and we extend the approach originally introduced by Cook and Zhu (2009) and Despotis et al. (2010) to deal with non-linear value function in DEA.

The rest of the paper unfolds as follows. In Section 2, we provide a brief description of the Almeida and Dias (2012) motivating approach. In Section 3, we develop our alternative modelling approach to value-based DEA. In Section 4, we revisit the case originally studied in the aforementioned work (Almeida and Dias, 2012) and we provide and compare the results obtained in the light of our approach. In section five we provide our main conclusions.

2 An additive DEA model incorporating value preferences

In Almeida and Dias (2012), preference elicitation protocols drawn from the MCDA are used in the frame of weighted additive DEA model, as a mean to incorporate user preferences in the DEA efficiency assessments. Below we outline their approach.

Consider n DMUs that use m inputs to produce s outputs, where y_{rj} denotes the level of the output r ($r = 1,\ldots,s$) produced by the DMU j ($j = 1,\ldots,n$), x_{ij} denotes the level of the input i ($i = 1,\ldots,m$) used by the DMU j. Let $Y_j = (y_{1j}, y_{2j},\ldots,y_{sj})$ and $X_j = (x_{1j}, x_{2j},\ldots,x_{mj})$ denote respectively the vectors of outputs and inputs for unit j. The Almeida and Dias (2012) approach unfolds in three phases:

2.1 Phase 1

The raw values of the observed inputs and outputs are mapped onto the value interval [0, 1]. That is, the inputs and the outputs, as measured in their original scales, are converted into a value scale, by assuming either linear or non-linear value functions V. By this transformation, all factors are treated as outputs to be maximised: $V_j(X_j, Y_j) = (v_{1j}, v_{2j},\ldots,v_{mj}, v_{m+1,j},\ldots, v_{m+s,j})$. The overall value of unit j is given in the additive form

$$U_j\left[V_j\left(X_j, Y_j\right)\right] = \sum_{k=1}^{m+s} v_{kj} w_k$$

The weights w_k, $k = 1,\ldots,m + s$ are dimensionless scaling constants. Optimal weights are calculated for each individual unit j in phase 2.

2.2 Phase 2

The following linear program is solved for a unit j_0 at a time:

$$\begin{aligned}
&\min \quad d \\
&s.t. \\
&\sum_{k=1}^{m+s} w_k v_{kj} - \sum_{k=1}^{m+s} w_k v_{kj_0} \le d \quad (j = 1,\ldots,n) \\
&\sum_{k=1}^{m+s} w_k = 1 \\
&\left(w_1, w_2,\ldots, w_{m+s}\right) \in W
\end{aligned} \tag{1}$$

where W denotes the set of intra-weight constraints reflecting the user's preferences. By convention, the weights are normalised so as to sum up to 1. Model (1) estimates for unit j_0 an optimal vector of weights $(w_1^{j_0}, w_2^{j_0}, \ldots, w_{m+s}^{j_0})$ that minimises, in the min-max sense, the loss of value to the best unit. Let d^{j_0} denote the optimal value of d in the optimal solution of (1). Then, if $d^{j_0} = 0$ and $w_k^{j_0} > 0, k = 1, \ldots, m+s$ for at least one optimal solution of (1), the unit j_0 is characterised as efficient. Otherwise, it is inefficient.

2.3 Phase 3

The following linear program is solved for every inefficient unit j_0 to find its projection on the efficient frontier:

$$\begin{aligned}
&\max \quad z = \sum_{k=1}^{m+s} w_k^{j_0} s_k \\
&s.t. \\
&\sum_{j=1}^{n} v_{kj} \lambda_j - s_k = v_{kj_0} \quad (k = 1, \ldots, m+s) \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0 \quad (j = 1, \ldots, n), \quad s_k \geq 0 \quad (k = 1, \ldots, m+s)
\end{aligned} \tag{2}$$

Model (2) is the envelopment form of a weighted additive DEA model, where only outputs are considered. For the optimal values d^{j_0} and z^{j_0} of the objective functions of models (1) and (2) holds that $d^{j_0} = z^{j_0}$.

As mentioned above, the weights w_k, $k = 1,\ldots,m + s$ are scaling constants, estimated for each unit at its best advantage in phase 2. In Almeida and Dias (2012) and Gouveia et al. (2008), these weights are generally interpreted as 'value trade-offs for the client'. To be exact, as long as each unit is left free to define its own (optimal) weights in phase 2, these value trade-offs differ from one unit to another and each time are estimated in favour of the evaluated unit. A limitation of this approach, which in fact is attributed to the choice of the additive DEA model, is that no direct measure of efficiency is provided. It only discriminates the efficient and the inefficient units. In our novel approach presented in the next section, the aforementioned weights acquire a particular meaning and are easily interpreted.

3 An alternative approach to value-based DEA

Consider n DMUs that use m inputs $(X_1, X_2,\ldots,X_m)$ to produce s outputs $(Y_1, Y_2,\ldots,Y_s)$. Given the output vector $Y_j = (y_{1j}, y_{2j},\ldots,y_{sj})$ of unit j, its overall value $U(Y_j)$ is given by the additive value function:

$$U(Y_j) = \sum_{r=1}^{s} U_r(y_{rj})$$

As long as the higher the levels of the outputs the greater their values, the partial value functions U_r, $r = 1,\dots,s$ are assumed non-decreasing. Notice that these partial value functions are generalisations of the so called *virtual outputs* in the DEA context, which are typically assumed linear, with the *total virtual output* given by

$$U\left(Y_j\right)=\sum_{r=1}^{s} y_{rj} u_r$$

where u_r, $r = 1,\dots,s$ are the weights assigned to the outputs. As concerns the inputs, we define the overall value $V(X_j)$ of the input vector $X_j = (x_{1j}, x_{2j},\dots,x_{mj})$ by

$$V\left(X_j\right)=\sum_{i=1}^{m} V_i\left(x_{ij}\right)$$

As the less the input level the highest its value, the partial value functions V_i, $i = 1,\dots,m$ of the individual inputs are assumed non-increasing. Notice, again, that in the original DEA models, the partial value functions of the inputs (*virtual inputs*) are assumed linear and the *total virtual input* is given by

$$V\left(X_j\right)=\sum_{i=1}^{m} x_{ij} v_i$$

where v_i, $i = 1,\dots,m$ are the weights assigned to the inputs. However, as $V(X_j)$ forms the denominator of the efficiency ratio, the individual virtual inputs are considered non-decreasing value functions, so as excess inputs are penalised. Assuming, for our developments, non-increasing value functions for the inputs allows us to treat the inputs as outputs. With such an arrangement, the value-based relative efficiency E_{j0} of the evaluated unit j_0 is estimated by the following general model:

$$\begin{aligned}
&\max E_{j0}=\sum_{r=1}^{s} U_r\left(y_{rj0}\right)+\sum_{i=1}^{m} V_i\left(x_{ij0}\right) \\
&s.t. \\
&\sum_{r=1}^{s} U_r\left(y_{rj}\right)+\sum_{i=1}^{m} V_i\left(x_{ij}\right)\leq 1 \quad (j=1,\dots,n)
\end{aligned} \tag{3}$$

Model (3) is equivalent to an input-oriented DEA model with $m + s$ outputs and a dummy input, set at the level of 1 for all the units. In a different context, this sort of a DEA-like model was introduced by Despotis (2005) as an index-maximising model for the reassessment of the human development index (HDI) via DEA. Unlike the basic assumption permeating the original DEA, that the virtual inputs and outputs are linear functions of the weights, we allow for non-linear value functions whenever necessary. Relaxing the linearity assumption, allows us to treat cases where, for example, the marginal value of an output diminishes as the output increases. We build to this end on Cook and Zhu (2009) and Despotis et al. (2010) who introduced piece-wise linear representations for non-linear virtual inputs and outputs in DEA.

3.1 Modelling the value functions

In general, the non-linearity requirement is desirable for particular outputs (inputs) only, with the rest of them complying with the linearity assumption. To distinguish them, let us call the former *non-linear (NL) outputs (inputs)* and the latter *linear (L) outputs (inputs)*. Without loss of generality, we assume that the first d ($r = 1,\dots,d$) outputs are linear and the rest of them (i.e., for $r = d + 1,\dots,s$) are non-linear. Analogously, the first t ($i = 1,\dots,t$) inputs are assumed linear and the rest of them ($i = t + 1,\dots,m$) are assumed non-linear.

3.1.1 Linear outputs

Let $l_r \le \min_j\{y_{rj}\}$ and $h_r \ge \max_j\{y_{rj}\}$ be fixed minimum and maximum values for output r, set so as the range $[l_r, h_r] \supseteq [\min_j\{y_{rj}\}, \max_j\{y_{rj}\}]$ covers the observed outputs of the entire set of units. By convention, we set $U_r(l_r) = 0$. Then, the value of any $y_{rj} \in [l_r, h_r]$ is given by:

$$U_r\left(y_{rj}\right) = \left(y_{rj} - l_r\right)u_r$$

Applying the following transformation:

$$U_r\left(y_{rj}\right) = \left(h_r - l_r\right)u_r \frac{y_{rj} - l_r}{h_r - l_r} = \hat{y}_{rj}\, p_r$$

we get the value of $y_{rj} \in [l_r, h_r]$ as function of the new variable p_r as:

$$U_r\left(y_{rj}\right) = \hat{y}_{rj}\, p_r \tag{4}$$

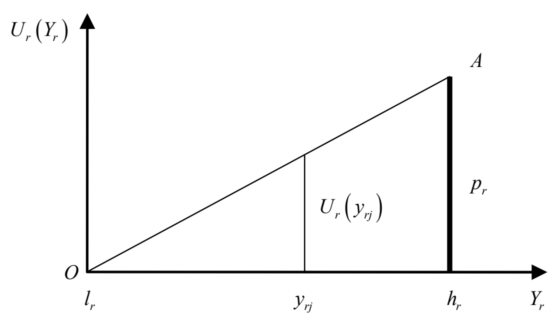
with

$$\hat{y}_{rj} = \frac{y_{rj} - l_r}{h_r - l_r}$$

From the above transformation derives that for any two output observations y_{rj} and y_{rk} holds that

$$y_{rj} \ge y_{rk} \Leftrightarrow U_r\left(y_{rj}\right) \ge U_r\left(y_{rk}\right)$$

Figure 1 Value function for a linear output measure Y_r



As depicted in Figure 1, the above transformation alters the weight variable u_r, which represents the slope of the line OA, to the new variable p_r that represents the value of h_r. The coefficient $\hat{y}_{rj}$ is now dimensionless and the term $\hat{y}_{rj} p_r$ represents the value of the output y_{rj} as a proportion of p_r.

3.1.2 Non-linear outputs

We model the non-decreasing value functions for the non-linear outputs in a piece-wise linear form. To this end, we assume $k_r + 1$ breakpoints that split the range $[l_r, h_r]$ of the non-linear output r in k_r segments: $[b_r^1, b_r^2], [b_r^2, b_r^3], \ldots, [b_r^{k_r}, b_r^{k_r+1}]$, with $b_r^1 = l_r$ and $b_r^{k_r+1} = h_r$. By convention, we set $U_r(l_r) = 0$. Then, any output $y_{rj} \in [l_r, h_r]$ can be decomposed as $y_{rj} = l_r + \delta_{rj}^1 + \delta_{rj}^2 + \cdots + \delta_{rj}^{k_r}$, where

$$\delta_{rj}^1 = \begin{cases} y_{rj} - b_r^1 & \text{if } y_{rj} \le b_r^2 \\ b_r^2 - b_r^1 & \text{if } y_{rj} > b_r^2 \end{cases}$$

$$\delta_{rj}^\mu = \begin{cases} 0 & \text{if } y_{rj} \le b_r^\mu \\ y_{rj} - b_r^\mu & \text{if } b_r^\mu < y_{rj} \le b_r^{\mu+1} \\ b_r^{\mu+1} - b_r^\mu & \text{if } y_{rj} > b_r^{\mu+1} \end{cases}, \quad \mu = 2, 3, \ldots, k_r - 1$$

$$\delta_{rj}^{k_r} = \begin{cases} 0 & \text{if } y_{rj} \le b_r^{k_r} \\ y_{rj} - b_r^{k_r} & \text{if } b_r^{k_r} < y_{rj} \le b_r^{k_r+1} \end{cases}$$

Assuming that the value function is linear in each segment, we assign a distinct weight variable $u_{r\mu}$ to each segment $\mu = 1, 2, \ldots, k_r$. Then, the partial value $U_r(y_{rj})$ is given in a piece-wise linear form as:

$$U_r\left(y_{rj}\right) = \delta_{rj}^1 u_{r1} + \delta_{rj}^2 u_{r2} + \cdots + \delta_{rj}^{k_r} u_{rk_r} = \sum_{\mu=1}^{k_r} \delta_{rj}^\mu u_{r\mu} \tag{5}$$

Applying to each segment the same transformation introduced for the linear outputs above, we get the value function (5) in terms of the new variables $p_{r1}, p_{r2}, \ldots, p_{rk_r}$ as follows:

$$U_r\left(y_{rj}\right) = \hat{\delta}_{rj}^1 p_{r1} + \hat{\delta}_{rj}^2 p_{r2} + \ldots + \hat{\delta}_{rj}^{k_r} p_{rk_r} = \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^\mu p_{r\mu} \tag{6}$$

with

$$\hat{\delta}_{rj}^\mu = \frac{\delta_{rj}^\mu}{b_r^{\mu+1} - b_r^\mu}, \quad \mu = 1, 2, \ldots, k_r$$

It is straightforward from (6) that $U_r\left(h_r\right) = p_{r1} + p_{r1} + \ldots + p_{rk_r}$.

Figure 2 Value function for a non-linear output measure Y_r

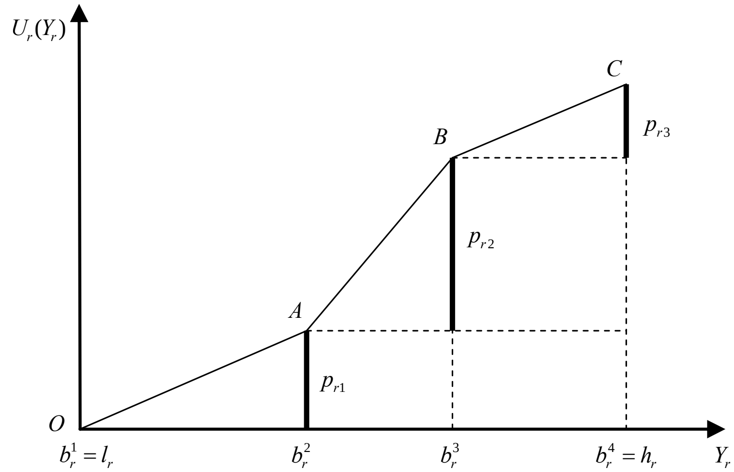


Figure 2 depicts a piece-wise linear value function for a non-linear output measure Y_r decomposed in three segments. With the above transformations, the weight variables u_{r1}, u_{r2} and u_{r3}, which represent respectively the slopes of the line segments OA, AB and BC, are replaced by the value variables p_{r1}, p_{r2} and p_{r3}, which represent the value increments in the intervals $[b_r^1, b_r^2], [b_r^2, b_r^3]$ and $[b_r^3, b_r^4]$ respectively.

Putting all together, i.e., the value functions of the linear and the non-linear outputs as given in (4) and (6) respectively, we get the value function (total virtual output) for the unit j, as follows:

$$U(Y_j) = \sum_{r=1}^{d} \hat{y}_{rj} p_r + \sum_{r=d+1}^{s} \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^{\mu} p_{r\mu} \tag{7}$$

In equation (7), the first summation refers to linear outputs, whereas the second summation refers to non-linear outputs.

3.1.3 Linear inputs

As mentioned at the beginning of the section, our value-based modelling approach assumes non-increasing value functions for the inputs as a means to treat the inputs as outputs. Let $l_i \leq \min_j \{x_{ij}\}$ and $h_i \geq \max_j \{x_{ij}\}$ be fixed minimum and maximum values for input i, set so as the range $[l_i, h_i] \supseteq [\min_j \{x_{ij}\}, \max_j \{x_{ij}\}]$ covers the observed inputs of the entire set of units. By convention, we set $V_i(h_i) = 0$. Then the value of any $x_{ij} \in [l_i, h_i]$ is given by:

$$V_i(x_{ij}) = (h_i - x_{ij}) v_i$$

Applying the transformation:

$$V_i\left(x_{ij}\right) = \left(h_i - l_i\right) v_i \frac{h_i - x_{ij}}{h_i - l_i} = \hat{x}_{ij} q_i$$

we get the value of $x_{ij} \in [l_i, h_i]$ as function of the new variable q_i as:

$$V_i\left(x_{ij}\right) = \hat{x}_{ij} q_i \tag{8}$$

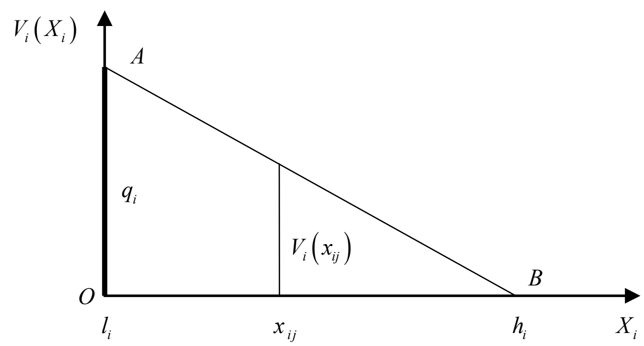
with

$$\hat{x}_{ij} = \frac{h_i - x_{ij}}{h_i - l_i}$$

From the above transformation derives that for any two input observations x_{ij} and x_{ik} holds that

$$x_{ij} \geq x_{ik} \Leftrightarrow V_i\left(x_{ij}\right) \leq V_i\left(x_{ik}\right)$$

Figure 3 Value function for a linear input measure X_i



As depicted in Figure 3, the above transformation alters the weight variable v_i, which represents the tangent of the angle $O\hat{B}A$, to the new variable q_i that represents the value of the most preferred input level l_i. The coefficient $\hat{x}_{ij}$ is now dimensionless and the term $\hat{x}_{ij} q_i$ represents the value of the output x_{ij} as a proportion of q_i.

3.1.4 Non-linear inputs

We model analogously non-increasing value functions for the non-linear inputs in a piece-wise linear form. Indeed, if $k_i + 1$ is the number of breakpoints that split the interval $[l_i, h_i]$ in k_i segments $[a_i^1, a_i^2], [a_i^2, a_i^3], \ldots, [a_i^{k_i}, a_i^{k_i+1}]$ with $a_i^1 = l_i$ and $a_i^{k_i+1} = h_i$, any input value $x_{ij} \in [l_i, h_i]$ is decomposed as $x_{ij} = h_i - (\gamma_{ij}^1 + \gamma_{ij}^2 + \ldots + \gamma_{ij}^{k_i})$ where:

$$\gamma_{ij}^{1}=\begin{cases}0 & \text{if } x_{ij}\geq a_i^2\\ a_i^2-x_{ij} & \text{if } a_i^1\leq x_{ij}<a_i^2\end{cases}$$

$$\gamma_{ij}^{\mu}=\begin{cases}0 & \text{if } x_{ij}\geq a_i^{\mu+1}\\ a_i^{\mu+1}-x_{ij} & \text{if } a_i^{\mu}\leq x_{ij}<a_i^{\mu+1}, \quad \mu=2,3,\ldots,k_i-1\\ a_i^{\mu+1}-a_i^{\mu} & \text{if } x_{ij}<a_i^{\mu}\end{cases} \tag{9}$$

$$\gamma_{ij}^{k_i}=\begin{cases}a_i^{k_i+1}-x_{ij} & \text{if } x_{ij}\geq a_i^{k_i}\\ a_i^{k_i+1}-a_i^{k_i} & \text{if } x_{ij}<a_i^{k_i}\end{cases}$$

Assigning a distinct weight variable $v_{i\mu}$ to each segment $\mu = 1, 2,\ldots,k_i$, the partial value $V_i(x_{ij})$ is given in a piece-wise linear form as:

$$V_i\left(x_{ij}\right)=\gamma_{ij}^1 v_{i1}+\gamma_{ij}^2 v_{i2}+\ldots+\gamma_{ij}^{k_i} v_{ik_i}=\sum_{\mu=1}^{k_i}\gamma_{ij}^{\mu} v_{i\mu} \tag{10}$$

Applying to each segment the same transformation introduced for the linear inputs, we get the value function (10) in terms of the new variables $q_{i1}, q_{i2},\ldots,q_{ik_i}$ as follows:

$$V_i\left(x_{ij}\right)=\hat{\gamma}_{ij}^1 q_{i1}+\hat{\gamma}_{ij}^2 q_{i2}+\ldots+\hat{\gamma}_{ij}^{k_i} q_{ik_i}=\sum_{\mu=1}^{k_i}\hat{\gamma}_{ij}^{\mu} q_{i\mu} \tag{11}$$

with

$$\hat{\gamma}_{ij}^{\mu}=\frac{\gamma_{ij}^{\mu}}{a_i^{\mu+1}-a_i^{\mu}}, \quad \mu=1,2,\ldots,k_i$$

It is straightforward from (11) that $V_i(l_i)=q_{i1}+q_{i2}+\ldots+q_{ik_i}$.

Figure 4 Value function for a non-linear input measure X_i

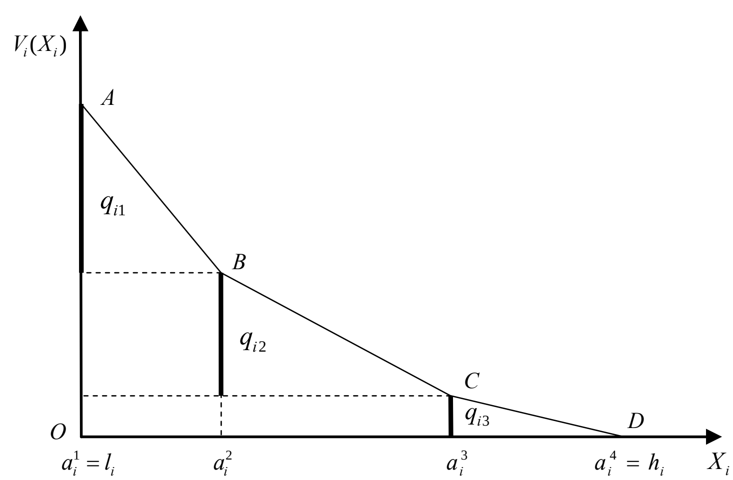


Figure 4 depicts a piece-wise linear value function for a non-linear input measure X_i decomposed in three segments. With the above transformations, the weight variables v_{i1}, v_{i2} and v_{i3}, which represent respectively the slopes of the line segments AB, BC and CD, are replaced by the value variables q_{i1}, q_{i2} and q_{i3}, which represent the value decrements in the intervals $[a_i^1, a_i^2], [a_i^2, a_i^3]$ and $[a_i^3, a_i^4]$ respectively.

Summing up (8) and (11), we get the value function (total virtual input) for the unit j, as follows:

$$V\left(X_j\right) = \sum_{i=1}^{t} \hat{x}_{ij} q_i + \sum_{i=t+1}^{m} \sum_{\mu=1}^{k_i} \hat{\gamma}_{ij}^{\mu} q_{i\mu} \tag{12}$$

In equation (12), the first summation refers to linear inputs, whereas the second summation refers to non-linear inputs.

3.2 Deriving the value-based DEA model

Putting together the value functions for outputs and inputs given in (7) and (12) respectively, we get the overall value E_j of the unit j as a function of the value variables p and q:

$$\begin{aligned} E_j(p, q) &= U\left(Y_j\right) + V\left(X_j\right) \\ &= \sum_{r=1}^{d} \hat{y}_{rj} p_r + \sum_{r=d+1}^{s} \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^{\mu} p_{r\mu} + \sum_{i=1}^{t} \hat{x}_{ij} q_i + \sum_{i=t+1}^{m} \sum_{\mu=1}^{k_i} \hat{\gamma}_{ij}^{\mu} q_{i\mu} \end{aligned} \tag{13}$$

The non-linear value functions in (13) can be customised so as to acquire specific properties on the basis of individual preferences. This can be done by introducing restrictions on the variables $p_{r\mu}$, $\mu = 1, 2, \ldots, k_r$ and $q_{i\mu}$, $\mu = 1, 2, \ldots, k_i$. For example, homogeneous restrictions of the form

$$\left(b_r^{\mu+2} - b_r^{\mu+1}\right) p_{r\mu} - c_{r\mu} \left(b_r^{\mu+1} - b_r^{\mu}\right) p_{r,\mu+1} \geq 0 \quad \left(c_{r\mu} \geq 1;\ \mu = 1, \ldots, k_r - 2\right)$$

impose concavity on the value function of output r, with the parameters $c_{r\mu}$ adjusting the sharpness of the diminishing returns. Analogously, the restrictions

$$\left(a_i^{\mu+2} - a_i^{\mu+1}\right) q_{i\mu} - z_{i\mu} \left(a_i^{\mu+1} - a_i^{\mu}\right) q_{i,\mu+1} \geq 0 \quad \left(z_{i\mu} \geq 1;\ \mu = 1, \ldots, k_i - 2\right)$$

impose convexity on the value function of input i.

Completing our developments, we provide below, with reference to the abstract model (3), our value-based model to assess the efficiency of the evaluated unit j_0:

$$\begin{aligned} &\max E_{j_0}(p, q) = \sum_{r=1}^{d} \hat{y}_{rjo} p_r + \sum_{r=d+1}^{s} \sum_{\mu=1}^{k_r} \hat{\delta}_{rj_o}^{\mu} p_{r\mu} + \sum_{i=1}^{t} \hat{x}_{ijo} q_i + \sum_{i=t+1}^{m} \sum_{\mu=1}^{k_i} \hat{\gamma}_{ijo}^{\mu} q_{i\mu} \\ &s.t. \\ &\sum_{r=1}^{d} \hat{y}_{rj} p_r + \sum_{r=d+1}^{s} \sum_{\mu=1}^{k_r} \hat{\delta}_{rj}^{\mu} p_{r\mu} + \sum_{i=1}^{t} \hat{x}_{ij} q_i + \sum_{i=t+1}^{m} \sum_{\mu=1}^{k_i} \hat{\gamma}_{ij}^{\mu} q_{i\mu} \leq 1 \quad j = 1, 2, \ldots, n \\ &p_r \geq 0 \quad (r = 1, \ldots, d) \end{aligned} \tag{14}$$

$$p_{r\mu} \geq 0 \quad (r = d+1,\ldots,s,\ \mu = 1,\ldots,k_r)$$
$$q_i \geq 0 \quad (i = 1,\ldots,t)$$
$$q_{i\mu} \geq 0 \quad (i = t+1,\ldots,m,\ \mu = 1,\ldots,k_i)$$
$$p_{r\mu},\ q_{i\mu} \in W$$

In the last constraint of (14), W denotes the region defined by user-specified restrictions on the variables that provide the non-linear value functions of outputs and inputs with properties reflecting the user's preferences.

The formulations presented above actually transform the original input/output dataset into an expanded dataset by decomposing each one of the non-linear inputs and outputs in auxiliary linear inputs and linear outputs respectively. This transformation allows us to perform the efficiency assessments without drawing away from the grounds of the DEA methodology. As a practical guide to implement the data transformation, one may consider in the set of units two dummy DMUs, one comprised by the fixed minimum values for the inputs and the outputs, the other comprised by the fixed maximum values. Notice here that these dummy units are not taken into account for the efficiency assessments. Then, the transformation is carried out in two steps: In the first step the non-linear inputs and outputs are decomposed on the basis of the segments assumed for each one of them to derive the expanded dataset. In a second step, the expanded data are normalised column-wise on the column ranges.

4 Application

We apply in this section our approach on a dataset drawn from Almeida and Dias (2012). The case concerns the efficiency assessment of 19 stores of a Portuguese retail chain in the pharmacy-cosmetics-hygiene sector. The five inputs considered, with their characterisations as linear (L) or non-linear (NL) and their scales of measurement, are: *average stock* (STK – L – €), *number of employees* (EMP – NL – full time equivalent), *salary costs* (SAC – L – €), *rent* (RNT – L – €) and *area* (ARE – NL – m^2). Two outputs are considered: *global sales* (SAL – L – €) and *family 4 sales/global sales* (F4 – NL – %). The input/output data are given in Table 1. In a preliminary stage, the raw data were originally transformed in values. Fixed minimum and maximum levels for the inputs and outputs were set, beyond the observed minima and maxima, as shown in Table 2.

Linear value functions were assumed for the three linear inputs X_{STK}, X_{SAC}, X_{RNT} and the linear output Y_{SAL}. The value functions of the non-linear input X_{ARE}, X_{EMP} and the non-linear output Y_{F4}, as shown in Figure 5, were obtained by interacting with the decision maker [see Almeida and Dias (2012) for details].

The efficiency estimates for the 19 DMUs, as given in Almeida and Dias (2012), are shown in the last column of Table 6 under the label z^*. They were obtained by solving models (1) and (2), with the phase 2 model (1) augmented with the following intra-weight constraints:

$$W = \begin{cases} w_{SAL} \geq w_{STK} \geq w_{RNT} \geq w_{SAC} \geq w_{F4} \geq w_{EMP} \geq w_{ARE} \\ w_{SAL} \leq 11.1 w_{ARE} \end{cases}$$

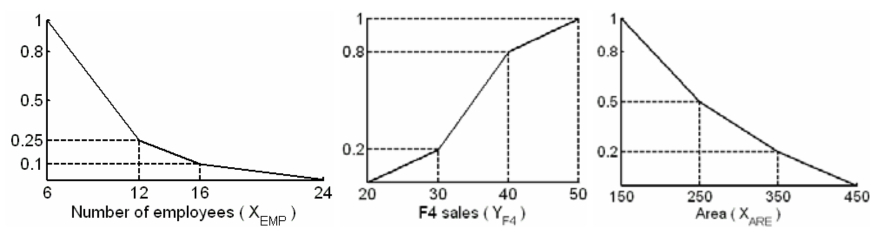
Table 1 Observed input/output data in original scales

DMU	*Inputs in original scales*					*Outputs in original scales*	
	X_{STK} *(L)*	X_{EMP} *(NL)*	X_{SAC} *(L)*	X_{RNT} *(L)*	X_{ARE} *(NL)*	Y_{SAL} *(L)*	Y_{F4} *(NL)*
1	360,614	13.2	153,071	275,240	213	1,994,652	36.4
2	263,736	9.5	111,409	117,916	213	1,194,289	44.6
3	628,938	17.8	218,122	492,305	436	3,841,226	32.2
4	479,582	16.5	189,495	134,824	262	2,299,879	23.8
5	600,449	15.9	222,567	411,982	331	3,905,880	39.7
6	299,876	12.3	159,338	185,368	208	1,554,821	37.2
7	171,010	9.2	92,436	124,355	231	625,315	22.0
8	354,506	13.9	153,228	231,525	400	1,570,432	24.4
9	521,819	13.1	155,918	145,527	222	2,249,522	28.0
10	357,204	7.3	96,041	179,931	200	1,505,312	45.4
11	307,347	11.3	135,895	171,760	313	1,387,585	28.1
12	701,109	15.8	214,814	300,106	290	5,425,809	32.4
13	392,894	15.2	170,675	250,726	216	2,269,410	40.5
14	604,291	20.0	222,424	387,543	443	3,410,820	27.8
15	272,851	12.0	148,268	159,532	197	1,410,839	33.9
16	327,304	11.9	181,352	168,006	207	1,263,137	29.1
17	356,157	11.5	130,337	181,693	286	1,371,183	26.5
18	152,850	11.4	87,223	147,252	301	877,671	32.6
19	295,598	13.3	193,606	160,607	199	1,634,121	26.0

Table 2 Fixed minimum and maximum levels for inputs and outputs

Fixed min/max levels	X_{STK} *(L)*	X_{EMP} *(NL)*	X_{SAC} *(L)*	X_{RNT} *(L)*	X_{ARE} *(NL)*	Y_{SAL} *(L)*	Y_{F4} *(NL)*
l	100,000	6	50,000	100,000	150	500,000	20
h	1,000,000	24	250,000	500,000	450	6,000,000	50

Figure 5 Piece-wise linear value functions for X_{EMP}, Y_{F4} and X_{ARE}



The ordinal constraints derived by ranking the factors, whereas the last constraint provides a trade-off between the most and the least important factors. The latter constraint was introduced to avoid null weights (Almeida and Dias, 2012).

Table 3 Expanded dataset in original scales

DMU	Inputs									Outputs			
	X_{STK} (L)	X_{EMP} (NL)			X_{SAC}(L)	X_{RNT}(L)	X_{ARE} (NL)			Y_{SAL} (L)	Y_{F4} (NL)		
		γ^1_{EMP}	γ^2_{EMP}	γ^3_{EMP}			γ^1_{ARE}	γ^2_{ARE}	γ^3_{ARE}		δ^1_{F4}	δ^2_{F4}	δ^3_{F4}
1	360,614	0.00	2.80	8.00	153,071	275,240	37.00	100.00	100.00	1,994,652	10.00	6.40	0.00
2	263,736	2.50	4.00	8.00	111,409	117,916	37.00	100.00	100.00	1,194,289	10.00	10.00	4.60
3	628,938	0.00	0.00	6.20	218,122	492,305	0.00	0.00	14.00	3,841,226	10.00	2.20	0.00
4	479,582	0.00	0.00	7.50	189,495	134,824	0.00	88.00	100.00	2,299,879	3.80	0.00	0.00
5	600,449	0.00	0.10	8.00	222,567	411,982	0.00	19.00	100.00	3,905,880	10.00	9.70	0.00
6	299,876	0.00	3.70	8.00	159,338	185,368	42.00	100.00	100.00	1,554,821	10.00	7.20	0.00
7	171,010	2.80	4.00	8.00	92,436	124,355	19.00	100.00	100.00	625,315	2.00	0.00	0.00
8	354,506	0.00	2.10	8.00	153,228	231,525	0.00	0.00	50.00	1,570,432	4.40	0.00	0.00
9	521,819	0.00	2.90	8.00	155,918	145,527	28.00	100.00	100.00	2,249,522	8.00	0.00	0.00
10	357,204	4.70	4.00	8.00	96,041	179,931	50.00	100.00	100.00	1,505,312	10.00	10.00	5.40
11	307,347	0.70	4.00	8.00	135,895	171,760	0.00	37.00	100.00	1,387,585	8.10	0.00	0.00
12	701,109	0.00	0.20	8.00	214,814	300,106	0.00	60.00	100.00	5,425,809	10.00	2.40	0.00
13	392,894	0.00	0.80	8.00	170,675	250,726	34.00	100.00	100.00	2,269,410	10.00	10.00	0.50
14	604,291	0.00	0.00	4.00	222,424	387,543	0.00	0.00	7.00	3,410,820	7.80	0.00	0.00
15	272,851	0.00	4.00	8.00	148,268	159,532	53.00	100.00	100.00	1,410,839	10.00	3.90	0.00
16	327304	0.10	4.00	8.00	181,352	168,006	43.00	100.00	100.00	1,263,137	9.10	0.00	0.00
17	356,157	0.50	4.00	8.00	130,337	181,693	0.00	64.00	100.00	1,371,183	6.50	0.00	0.00
18	152,850	0.60	4.00	8.00	87,223	147,252	0.00	49.00	100.00	877,671	10.00	2.60	0.00
19	295,598	0.00	2.70	8.00	193,606	160,607	51.00	100.00	100.00	1,634,121	6.00	0.00	0.00
Associated weight variables	v_{STK}	$v_{EMP,1}$	$v_{EMP,2}$	$v_{EMP,3}$	v_{SAC}	v_{RNT}	$v_{ARE,1}$	$v_{ARE,2}$	$v_{ARE,3}$	u_{SAL}	$u_{F4,1}$	$u_{F4,2}$	$u_{F4,3}$

Table 4 Breakpoints for the non-linear inputs and outputs in original scales

	Breakpoints for X_{EMP}				*Breakpoints for X_{ARE}*				*Breakpoints for Y_{F4}*			
	a^1_{EMP}	a^2_{EMP}	a^3_{EMP}	a^4_{EMP}	a^1_{ARE}	a^2_{ARE}	a^3_{ARE}	a^4_{ARE}	b^1_{F4}	b^2_{F4}	b^3_{F4}	b^4_{F4}
X_{EMP}	6	12	16	20								
X_{ARE}					150	250	350	450				
Y_{F4}									20	30	40	50

Table 5 Transformed dataset

DMU	X_{STK} (L)	X_{EMP} (NL)			X_{SAC} (L)	X_{RNT} (L)	X_{ARE} (NL)			Y_{SAL} (L)	*M* $Y_{\%F4}$ (NL)		
	$\hat{x}_{STK}$	$\hat{\gamma}^1_{EMP}$	$\hat{\gamma}^2_{EMP}$	$\hat{\gamma}^3_{EMP}$	$\hat{x}_{SAC}$	$\hat{x}_{RNT}$	$\hat{\gamma}^1_{ARE}$	$\hat{\gamma}^2_{ARE}$	$\hat{\gamma}^3_{ARE}$	$\hat{y}_{SAL}$	$\hat{\delta}^1_{F4}$	$\hat{\delta}^2_{F4}$	$\hat{\delta}^3_{F4}$
1	0.71	0.00	0.70	1.00	0.48	0.56	0.37	1.00	1.00	0.27	1.00	0.64	0.00
2	0.82	0.42	1.00	1.00	0.69	0.96	0.37	1.00	1.00	0.13	1.00	1.00	0.46
3	0.41	0.00	0.00	0.78	0.16	0.02	0.00	0.00	0.14	0.61	1.00	0.22	0.00
4	0.58	0.00	0.00	0.94	0.30	0.91	0.00	0.88	1.00	0.33	0.38	0.00	0.00
5	0.44	0.00	0.02	1.00	0.14	0.22	0.00	0.19	1.00	0.62	1.00	0.97	0.00
6	0.78	0.00	0.93	1.00	0.45	0.79	0.42	1.00	1.00	0.19	1.00	0.72	0.00
7	0.92	0.47	1.00	1.00	0.79	0.94	0.19	1.00	1.00	0.02	0.20	0.00	0.00
8	0.72	0.00	0.53	1.00	0.48	0.67	0.00	0.00	0.50	0.19	0.44	0.00	0.00
9	0.53	0.00	0.73	1.00	0.47	0.89	0.28	1.00	1.00	0.32	0.80	0.00	0.00
10	0.71	0.78	1.00	1.00	0.77	0.80	0.50	1.00	1.00	0.18	1.00	1.00	0.54
11	0.77	0.12	1.00	1.00	0.57	0.82	0.00	0.37	1.00	0.16	0.81	0.00	0.00
12	0.33	0.00	0.05	1.00	0.18	0.50	0.00	0.60	1.00	0.90	1.00	0.24	0.00
13	0.67	0.00	0.20	1.00	0.40	0.62	0.34	1.00	1.00	0.32	1.00	1.00	0.05
14	0.44	0.00	0.00	0.50	0.14	0.28	0.00	0.00	0.07	0.53	0.78	0.00	0.00
15	0.81	0.00	1.00	1.00	0.51	0.85	0.53	1.00	1.00	0.17	1.00	0.39	0.00
16	0.75	0.02	1.00	1.00	0.34	0.83	0.43	1.00	1.00	0.14	0.91	0.00	0.00
17	0.72	0.08	1.00	1.00	0.60	0.80	0.00	0.64	1.00	0.16	0.65	0.00	0.00
18	0.94	0.10	1.00	1.00	0.81	0.88	0.00	0.49	1.00	0.07	1.00	0.26	0.00
19	0.78	0.00	0.68	1.00	0.28	0.85	0.51	1.00	1.00	0.21	0.60	0.00	0.00
Associated value variables	q_{STK}	$q_{EMP,1}$	$q_{EMP,2}$	$q_{EMP,3}$	q_{SAC}	q_{RNT}	$q_{ARE,1}$	$q_{ARE,2}$	$q_{ARE,3}$	p_{SAL}	$p_{F4,1}$	$p_{F4,2}$	$p_{F4,3}$

Applying our data transformation-variable alteration technique discussed in the previous section, we obtain the expanded dataset and its range-normalised counterpart, as shown in Table 3 and Table 5 respectively. For comparison purposes, we assumed the same fixed minimum and maximum values shown in Table 2. The breakpoints for the non-linear factors are set to the values originally considered in Almeida and Dias (2012), as shown in Table 4.

To build our model for the specific dataset, without drawing away from the preferential information assumed in the original work, we make the following adjustments that imitate the same decision situation.

4.1 Value functions

To maintain the preferential information assumed for the non-linear inputs and outputs we introduce in our model the following constraints in terms of the value variables:

$$
\begin{aligned}
&q_{EMP,1} - 5q_{EMP,2} = 0 \\
&2q_{EMP,2} - 3q_{EMP,3} = 0 \\
&3q_{ARE,1} - 5q_{ARE,2} = 0 \\
&2q_{ARE,2} - 3q_{ARE,3} = 0 \\
&3p_{F4,1} - p_{F4,2} = 0 \\
&p_{F4,2} - 3p_{F4,3} = 0
\end{aligned}
$$

For example, as concerns the non-linear output $F4$, the slopes of the line segments of the value function are (see Figure 5):

$$u_{F4,1} = 0.02, u_{F4,2} = 0.06, u_{F4,3} = 0.02$$

As the ratio of these weights is of interest in our model, we apply the following variable transformation to derive these ratios in terms of the corresponding value variables:

$$
\begin{aligned}
&\frac{u_{F4,1}}{u_{F4,2}} = \frac{1}{3} \Leftrightarrow \frac{u_{F4,1}\left(b_{F4}^{2} - b_{F4}^{1}\right)}{u_{F4,2}\left(b_{F4}^{3} - b_{F4}^{2}\right)} = \frac{b_{F4}^{2} - b_{F4}^{1}}{3\left(b_{F4}^{3} - b_{F4}^{2}\right)} \Leftrightarrow \frac{p_{F4,1}}{p_{F4,2}} = \frac{10}{3(40-30)} = \frac{1}{3} \\
&\frac{u_{F4,2}}{u_{F4,3}} = 3 \Leftrightarrow \frac{u_{F4,2}\left(b_{F4}^{3} - b_{F4}^{2}\right)}{u_{F4,3}\left(b_{F4}^{4} - b_{F4}^{3}\right)} = \frac{3\left(b_{F4}^{3} - b_{F4}^{2}\right)}{\left(b_{F4}^{4} - b_{F4}^{3}\right)} \Leftrightarrow \frac{p_{F4,2}}{p_{F4,3}} = \frac{3(40-30)}{(50-40)} = 3
\end{aligned}
$$

4.2 Ranking and trade-off constraints

Analogously, the ordinal and trade-off constraints W assumed in the original work (Almeida and Dias, 2012) are translated in terms of our model as follows:

$$
\begin{aligned}
&p_{SAL} \geq q_{STK} \geq q_{RNT} \geq q_{SAC} \geq p_{F4,1} + p_{F4,2} + p_{F4,3} \\
&\geq q_{EMP,1} + q_{EMP,2} + q_{EMP,3} \geq q_{ARE,1} + q_{ARE,2} + q_{ARE,3} \\
&p_{SAL} \leq 11.1\left(q_{ARE,1} + q_{ARE,2} + q_{ARE,3}\right)
\end{aligned}
$$

On the basis of the above adjustments, the model (14) that assesses the relative efficiency of the evaluated unit j_0 takes the following specific form:

$$\max E_{j_o} = \left[\begin{array}{l} \hat{y}_{SAL,j_0} p_{SAL} + \hat{\delta}^1_{F4,j_0} p_{F4,1} + \hat{\delta}^2_{F4,j_0} p_{F4,2} + \hat{\delta}^3_{F4,j_0} p_{F4,3} + \hat{x}_{STK,j_0} q_{STK} \\ +\hat{\gamma}^1_{EMP,j_0} q_{EMP,1} + \hat{\gamma}^2_{EMP,j_0} q_{EMP,2} + \hat{\gamma}^3_{EMP,j_0} q_{EMP,3} + \hat{x}_{SAC,j_0} q_{SAC} \\ +\hat{x}_{RNT,j_0} q_{RNT} + \hat{\gamma}^1_{ARE,j_0} q_{ARE,1} + \hat{\gamma}^2_{ARE,j_0} q_{ARE,2} + \hat{\gamma}^3_{ARE,j_0} q_{ARE,3} \end{array} \right.$$

$s.t.$

[section 1]

$$\begin{array}{l} \hat{y}_{SAL,j} p_{SAL} + \hat{\delta}^1_{F4,j} p_{F4,1} + \hat{\delta}^2_{F4,j} p_{F4,2} + \hat{\delta}^3_{F4,j} p_{F4,3} + \hat{x}_{STK,j} q_{STK} \\ +\hat{\gamma}^1_{EMP,j} q_{EMP,1} + \hat{\gamma}^2_{EMP,j} q_{EMP,2} + \hat{\gamma}^3_{EMP,j} q_{EMP,3} + \hat{x}_{SAC,j} q_{SAC} \\ +\hat{x}_{RNT,j} q_{RNT} + \hat{\gamma}^1_{ARE,j} q_{ARE,1} + \hat{\gamma}^2_{ARE,j} q_{ARE,2} + \hat{\gamma}^3_{ARE,j} q_{ARE,3} \le 1 \quad (j = 1, \ldots n) \end{array}$$

[section 2]

$$\begin{array}{l} q_{EMP,1} - 5q_{EMP,2} = 0 \\ 2q_{EMP,2} - 3q_{EMP,3} = 0 \\ 3q_{ARE,1} - 5q_{ARE,2} = 0 \\ 2q_{ARE,2} - 3q_{ARE,3} = 0 \\ 3p_{F4,1} - p_{F4,2} = 0 \\ p_{F4,2} - 3p_{F4,3} = 0 \end{array}$$

[section 3]

$$\begin{array}{l} p_{SAL} - q_{STK} \ge 0 \\ q_{STK} - q_{RNT} \ge 0 \\ q_{RNT} - q_{SAC} \ge 0 \\ q_{SAC} - p_{F4,1} - p_{F4,2} - p_{F4,3} \ge 0 \\ p_{F4,1} + p_{F4,2} + p_{F4,3} - q_{EMP,1} - q_{EMP,2} - q_{EMP,3} \ge 0 \\ q_{EMP,1} + q_{EMP,2} + q_{EMP,3} - q_{ARE,1} - q_{ARE,2} - q_{ARE,3} \ge 0 \\ p_{SAL} - 11.1q_{ARE,1} - 11.1q_{ARE,2} - 11.1q_{ARE,3} \le 0 \end{array} \tag{15}$$

$$p_{(.)}, q_{(.)} \ge 0$$

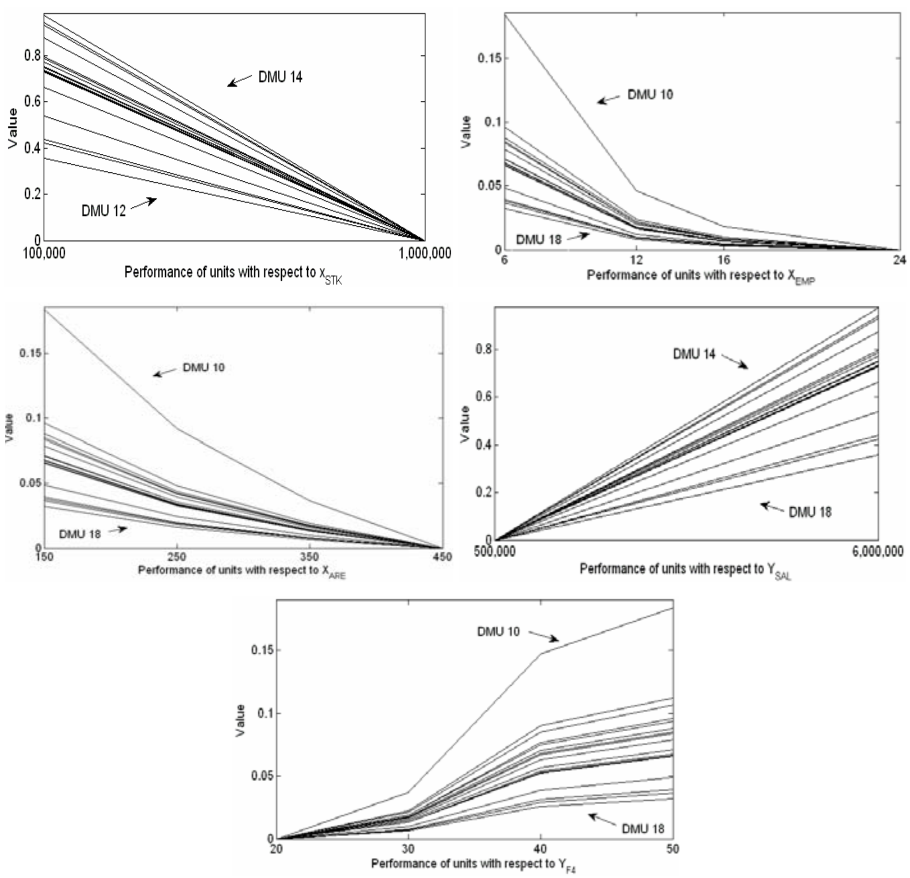
The [section 1] comprises the ordinary DEA constraints. The [section 2] constraints derive from the preferential information that drive the forms of the piece-wise linear value functions, whereas the [section 3] constraints are formed on the basis of the ranking and trade-off information assumed in the original study (Almeida and Dias, 2012). The efficiency scores and the optimal solutions (in terms of value variables) are presented in Table 6. Figure 6 exhibits the value functions assessed by the 19 DMUs for the inputs X_{STK}, X_{EMP}, $X_{ARE,}$ and the outputs Y_{SAL} and Y_{F4}.

Table 6 Efficiency scores and optimal solutions in terms of value variables

DMU	E_j	q_{STK}	$q_{EMP,1}$	$q_{EMP,2}$	$q_{EMP,3}$	q_{SAC}	q_{RNT}	$q_{ARE,1}$	$q_{ARE,2}$	$q_{ARE,3}$	p_{SAL}	$p_{F4,1}$	$p_{F4,2}$	$p_{F4,3}$	z^*
1	0.915	0.772	0.072	0.014	0.010	0.096	0.096	0.048	0.029	0.019	0.772	0.019	0.058	0.019	0.046
2	1.000	0.423	0.029	0.006	0.004	0.093	0.423	0.019	0.011	0.008	0.423	0.019	0.056	0.019	0.000
3	0.789	0.930	0.063	0.013	0.008	0.084	0.084	0.042	0.025	0.017	0.930	0.017	0.050	0.017	0.118
4	0.901	0.469	0.049	0.010	0.007	0.066	0.469	0.033	0.020	0.013	0.730	0.013	0.039	0.013	0.057
5	0.888	0.796	0.054	0.011	0.007	0.112	0.112	0.036	0.022	0.014	0.796	0.022	0.067	0.022	0.061
6	0.937	0.736	0.050	0.010	0.007	0.066	0.192	0.033	0.020	0.013	0.736	0.013	0.040	0.013	0.035
7	0.995	0.358	0.027	0.005	0.004	0.358	0.358	0.018	0.011	0.007	0.358	0.007	0.022	0.007	0.003
8	0.805	0.876	0.059	0.012	0.008	0.149	0.149	0.039	0.024	0.016	0.876	0.016	0.047	0.016	0.106
9	0.896	0.472	0.050	0.010	0.007	0.066	0.472	0.033	0.020	0.013	0.734	0.013	0.040	0.013	0.060
10	1.000	0.184	0.138	0.028	0.018	0.184	0.184	0.092	0.055	0.037	0.661	0.037	0.110	0.037	0.000
11	0.876	0.788	0.053	0.011	0.007	0.071	0.206	0.035	0.021	0.014	0.788	0.014	0.043	0.014	0.067
12	1.000	0.085	0.064	0.013	0.008	0.085	0.085	0.042	0.025	0.017	0.942	0.017	0.051	0.017	0.000
13	0.937	0.754	0.051	0.010	0.007	0.107	0.107	0.034	0.020	0.014	0.754	0.021	0.064	0.021	0.034
14	0.753	0.974	0.066	0.013	0.009	0.088	0.088	0.044	0.026	0.018	0.974	0.018	0.053	0.018	0.137
15	0.947	0.729	0.049	0.010	0.007	0.066	0.190	0.033	0.020	0.013	0.729	0.013	0.039	0.013	0.029
16	0.859	0.540	0.036	0.007	0.005	0.049	0.540	0.024	0.015	0.010	0.540	0.010	0.029	0.010	0.080
17	0.841	0.439	0.030	0.006	0.004	0.391	0.439	0.020	0.012	0.008	0.439	0.008	0.024	0.008	0.089
18	1.000	0.357	0.024	0.005	0.003	0.357	0.357	0.016	0.010	0.006	0.357	0.006	0.019	0.006	0.000
19	0.920	0.750	0.051	0.010	0.007	0.068	0.196	0.034	0.020	0.014	0.750	0.014	0.041	0.014	0.044

The results obtained by our approach are straightly comparable with those given in Almeida and Dias (2012). Indeed, as shown in the second and the last columns of Table 6, exactly the same units (namely, the units 2, 10, 12 and 18) are estimated efficient with both approaches. From a computational burden aspect, although our linear program (15) is a little larger than the phase 2 and phase 3 programs (1) and (2) due to the additional variables derived from the segmentation of the non-linear factors and the associated [section 2]-constraints, it is only solved once for each unit. Recall here that according to the Almeida and Dias (2012) procedure, a phase 2 program is solved for each unit and then a phase 3 program is solved for each inefficient unit. In particular and in the context of their study, our program (14) is solved 19 times, whereas 34 runs are needed (19 for phase 2 and 15 for phase 3) to complete the assessments with their approach. Moreover, any ordinary ready-made DEA software is sufficient to solve our model, which is not the case for the three-phase procedure described in section 2.

Figure 6 Value functions assessed by the DMUS for X_{STK}, X_{EMP}, X_{ARE}, Y_{SAL} and Y_{F4}



5 Conclusions

We presented in this paper an alternative value-based DEA approach to assess the efficiency of DMUs on the basis of individual preferences. Expanding and then normalising the original input/output dataset on the column ranges is followed by a variable transformation that alters the original weight variables to value variables. This particular transformation is the key that enables the development of our value-based DEA model. Where non-linear value functions should be assumed, they are approximated by piece-wise linear value functions. There are two potential advantages of our approach when compared to the approach of Almeida and Dias (2012) that motivated our developments: it provides a measure of efficiency in the form of a ratio rather than in the form of a min-max loss of value; it requires fewer linear programs to be solved. The application of our approach to the case studied in Almeida and Dias (2012) showed that our results are straightly comparable to the original ones, when imitating the same decision situation.

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